Regularity of invariant distributions

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Abstract

We consider a discrete subgroup Γ of the isometry group G of the real hyperbolic space $X \cong H^n$ of dimension $n \geq 2$. We assume that Γ admits a fundamental domain with finitely many totally geodesic faces (i.e. Γ is geometrically finite). This domain may have finitely many cusps of rank varying in $\{1, \dots, n-1\}$. Γ acts by conformal transformations on the sphere $\partial X \cong S^{n-1}$ at infinity. By Λ_Γ we denote its limit set which is a closed Γ-invariant subset of ∂X of Hausdorff dimension $\dim_H(\Lambda_\Gamma)$. We consider a G-equivariant irreducible complex vector bundle $V \to \partial X$. Let Λ be the bundle of densities on ∂X. Then we define a real number $s(V)$ by the condition that $V \otimes \Lambda^{s(V)} \cong V^\sharp \otimes \Lambda$ as G-equivariant bundles, where $V^\sharp$ is the hermitian dual of V. The goal of the paper is to find lower bounds on the regularity of Γ-invariant distribution sections of V which are strongly supported on the limit set. In order to quantify this regularity we consider the spaces of sections of regularity $H^{p,r}$, $p \in (1, \infty)$, $r \in \mathbb{R}$. For $r \in \mathbb{N}_0$ the space $H^{p,r}$ is the space of distribution sections which have all their derivatives up to order r in L^p. For $r \in [0, \infty)$ we extend this scale by complex interpolation, while for negative r we define these spaces by duality. Our main result is the following.

Let

$$\begin{aligned} r^0 &:= (n-1)\frac{s(V)-1}{2} + \frac{n-1-\dim_H(\Lambda_\Gamma)}{p} \\ r &:= \min\left(r^0, (n-1)s(V) + \frac{n-1}{p} - \max_{cusps\ c} \mathrm{rank}(c)\right) \ , \end{aligned}$$

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where the maximum over the empty set is, by convention, equal to $-\infty$*. Moreover,* $\dim_H(\Lambda_\Gamma)$ *has to be replaced by half of the rank of the cusp if* Γ *is elementary parabolic. If* ϕ *is a* Γ*-invariant distribution section of* V *which is strongly supported on the limit set (and a cusp form), then* $\phi \in H^{p,s}$ *for all* $s < r$ $(s < r^0)$*.*

If Γ is convex cocompact, then strongly supported on the limit set just means supported on the limit set as a distribution (and the cusp form condition holds true automatically). But if Γ has cusps, then this condition is stronger. Generically, boundary values of cusp forms and residues of Eisenstein series have this property. Actually, in order to prove our result, we also need some additional technical assumptions.

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1 Statement of the result

1.1 Introduction and statement of the main theorem

Let G be one of the groups $Spin(1,n)$, $SO(1,n)_0$, $2 \le n \in \mathbb{N}$. Then the symmetric space X of G is the real hyperbolic space of dimension n. Furthermore, let $\Gamma \subset G$ be a discrete subgroup. We assume that Γ is *geometrically finite*. One of many equivalent formulations of this condition is that Γ admits a fundamental domain in X which has finitely many totally geodesic faces. In the following we assume that Γ is *torsion-free*, but we shall see later that one can drop this assumption.

By ∂X we denote the geodesic boundary of X. It can be identified with the space of all minimal parabolic subgroups of G such that the minimal parabolic subgroup $P \subset G$ corresponds to its unique fixed point $\infty_P \in \partial X$. Since G acts transitively on the set of minimal parabolic subgroups we can identify $\partial X \cong G/P$ for any fixed P.

The G-homogeneous complex vector bundles on ∂X are in one-to-one correspondence with finite-dimensional complex representations of P. On the one hand if (θ, V_θ) is a finite dimensional complex representation of P, then we can form the associated bundle $V(\theta) := G \times_{P,\theta} V_\theta$. On the other hand, if $V \to \partial X$ is a G-homogeneous complex vector bundle, then we let V_θ be the representation of P on the fibre of V over ∞_P, and we have a natural isomorphism $V(\theta) \cong V$.

In order to parametrize the irreducible representations of P we consider the exact sequence

$$0 \to N \to P \to L \to 0 \ ,$$

where N is the unipotent radical of P. Any finite-dimensional irreducible representation of P factors over the quotient L. This group fits into a natural exact sequence

$$0 \to M \to L \to A \to 0 \ , \tag{1}$$

where M is a compact group isomorphic to $Spin(n-1)$ or $SO(n-1)$, and A is isomorphic to the multiplicative group $\mathbb{R}^+$. Let $\mathfrak{a}$ denote the Lie algebra of A and $\mathfrak{a}^*_\mathbb{C}$ be the complexification of its dual.

For $\lambda \in \mathfrak{a}^*_\mathbb{C}$ we form the character $a \mapsto a^\lambda := \exp(\lambda(\log(a)))$. The group L acts on the Lie algebra $\mathfrak{n}$ of N. The induced representation on $\Lambda^{\dim(\mathfrak{n})}\mathfrak{n}$ factors over A and corresponds to the

character $2\rho \in \mathfrak{a}^*$. Furthermore, we define the element $\alpha \in \mathfrak{a}^*$ by

$$\alpha := \frac{2\rho}{\dim(\mathfrak{n})}$$

which plays the role of a unit.

If (θ, V_θ) is any irreducible representation of L, then its restriction to M is an irreducible unitary representation. Furthermore, the representation $\Lambda^{\dim(V_\theta)}\theta$ factors over A and corresponds to an element of $\mathfrak{a}^*_{\mathbb{C}}$ (except in the case $G = SL(2,\mathbb{R})$. In this case M may act non-trivially on $\Lambda^{\dim(V_\theta)}\theta$ and one employs the canonical split of (1) in order to define the character of A corresponding to θ).

Definition 1.1 *Given $(\sigma, V_\sigma) \in \hat{M}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we let $(\sigma_\lambda, V_{\sigma_\lambda})$ be the unique irreducible representation of P such that $(\sigma_\lambda)_{|M} = \sigma$ and $\Lambda^{\dim V_\sigma}\sigma_\lambda$ corresponds to $\dim(V_\sigma)(\rho - \lambda) \in \mathfrak{a}^*_{\mathbb{C}}$.*

The bundle of densities Λ on ∂X is given in this parametrization by $V(1_{-\rho})$, where $1 \in \hat{M}$ denotes the trivial representation. The bundle $V(\sigma_\lambda)^\sharp \otimes \Lambda$ is isomorphic to $V(\sigma_{-\bar{\lambda}})$ as G-homogeneous bundle. Note that $V(1_{-\rho})^s \otimes V(\sigma_\lambda) = V(\sigma_{\lambda-2s\rho})$. We conclude that

$$s(V(\sigma_\lambda)) = \frac{\mathrm{Re}(\lambda)}{\rho} ,$$

where $s(V(\sigma_\lambda))$ is the number introduced in the abstract.

Another important element of $\mathfrak{a}^*$ is the critical exponent δ_Γ of Γ which we now define. We fix any maximal compact subgroup $K \subset G$. This choice induces an embedding of L into P such that L is stable under the Cartan involution θ_K associated with K. Furthermore, we obtain a decomposition $L = MA$, where $M = L^{\theta_K}$ is the set of fixed points of θ_K, and if $a \in A$, then $\theta_K(a) = a^{-1}$. In particular we now can consider A as a subgroup of G. The character ρ fixes a semigroup $A_+ := \{a \in A \mid a^\rho \geq 1\}$. By the Cartan decomposition any element $g \in G$ can be written as $g = k_g a_g h_g$, where $k_g, h_g \in K$, and $a_g \in A_+$ is uniquely determined.

Definition 1.2 *The* critical exponent $\delta_\Gamma \in \mathfrak{a}^*$ *of Γ is defined as the infimum of the set $\{\mu \in \mathfrak{a}^* \mid \sum_{\gamma\in\Gamma} a_\gamma^{-\mu-\rho} < \infty\}$.*

Let $\partial X = \Omega_\Gamma \cup \Lambda_\Gamma$ be the Γ-invariant decomposition into the limit set Λ_Γ and the ordinary set Ω_Γ. The limit set Λ_Γ is closed and can be defined as the set of accumulation points of any

Γ-orbit in X. By results of Patterson [10] in case $n = 2$ and Sullivan [14] for all $n \geq 2$ the Hausdorff dimension of Λ_Γ is given by $\dim_H(\Lambda_\Gamma) = \frac{\delta_\Gamma + \rho}{\alpha}$ unless Γ is elementary parabolic.

Recall that a parabolic subgroup $P \subset G$ is called Γ-cuspidal if $\Gamma \cap P$ is infinite and projects to a precompact subset $l(\Gamma \cap P) \subset L$, where $l : P \to L$ is the projection. We form the compact group $M_\Gamma := \overline{l(\Gamma \cap P)} \subset M$. Furthermore, there exists a subgroup $N_\Gamma \subset N$ and a Langlands decomposition $P = MAN$ (choice of K) such that $\Gamma_P := \Gamma \cap P$ is a cocompact lattice in the unimodular group $P_\Gamma := N_\Gamma M_\Gamma$.

A *cusp* of Γ is a Γ-conjugacy class $[P]_\Gamma$ of Γ-cuspidal parabolic subgroups. The dimension $\dim(N_\Gamma)$ is called *the rank of the cusp* $\mathrm{rank}([P]_\Gamma)$ defined by P. We further define $\rho_{\Gamma_P} := \frac{1}{2}\mathrm{rank}([P]_\Gamma)\alpha$ and $\rho^{\Gamma_P} := \rho - \rho_{\Gamma_P}$. If $\dim(N_\Gamma) \in \{1, \ldots, \dim(\mathfrak{n}) - 1\}$, then we say that the cusp has *smaller rank*. If $\dim(N_\Gamma) = \dim(\mathfrak{n})$, then the cusp has *full rank*.

A finite-dimensional representation (φ, V_φ) of Γ is called *admissible twist* (shortly twist) if for any Γ-cuspidal parabolic subgroup P the representation φ extends to AP_Γ such that A acts by semisimple endomorphisms. For example, if φ is a finite-dimensional representation of G, then its restriction to Γ is an admissible twist.

Given a representation (θ, V_θ) of P and an admissible twist (φ, V_φ) of Γ we form the bundle $V(\theta, \varphi) = V(\theta) \otimes V_\varphi$. If π^θ denotes the left-regular representation of G on sections of $V(\theta)$, then we can form the representation $\pi^{\theta,\varphi} := \pi^\theta \otimes \varphi$ of Γ on sections of $V(\theta, \varphi)$. Note that $\pi^{\theta,\varphi}$ extends to AP_Γ for each Γ-cuspidal parabolic subgroup P.

Definition 1.3 *The* exponent δ_φ of the twist φ *is defined as the infimum of the set* $\{\mu \in \mathfrak{a}^* \mid \sup_{\gamma \in \Gamma} \|\varphi(\gamma)\| a_\gamma^{-\mu} < \infty\}$*, where* $\|.\|$ *is any norm on* $\mathrm{End}(V_\varphi)$*.*

Let $\tilde{\sigma}$ or $\tilde{\varphi}$ denote the $\mathbb{C}$-linear dual representations of σ or φ. The spaces

$$C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) := C^\infty(\partial X, V(\tilde{\sigma}_{-\lambda}, \tilde{\varphi}))^*$$

for varying $\lambda \in \mathfrak{a}^*_\mathbb{C}$ form locally trivial holomorphic bundles of dual Fréchet spaces so that it makes sense to speak of germs holomorphic families $(\phi_\lambda)_\lambda$ such that $\phi_\lambda \in C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$. Let ${}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ be the subspace of Γ-invariant distributions.

We call $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ *deformable*, if there is a germ at λ of a holomorphic family $(\phi_\mu)_\mu$ such that $\phi_\lambda = \phi$ and $\phi_\mu \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\mu, \varphi))$.

Definition 1.4 *Let* $\mathrm{Fam}_\Gamma(\sigma_\lambda,\varphi)\subset {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$ *be the subspace of all deformable* Γ*-invariant distributions.*

If $\phi\in\mathrm{Fam}_\Gamma(\sigma_\lambda,\varphi)$, then we say that ϕ is *strongly supported on the limit set* if $res^\Gamma(\phi)=0$. Here res^Γ is the restriction map introduced in [4] (we refer to Subsection 2.2 for details). More precisely, this condition means that there exists a germ at λ of a holomorphic family $(\phi_\mu)_\mu$, $\phi_\mu\in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\mu,\varphi))$, such that $\phi_\lambda=\phi$ and $(res^\Gamma\phi_\mu)_{|\mu=\lambda}=0$.

If ϕ is strongly supported on the limit set, then its support as a distribution $\mathrm{supp}(\phi)$ is contained in Λ_Γ. (In fact it is equal to Λ_Γ if $\sharp\Lambda_\Gamma\neq 2$.) The converse may be false if Γ has cusps.

Definition 1.5 *Let* $\mathrm{Fam}_\Gamma(\Lambda_\Gamma,\sigma_\lambda,\varphi)\subset\mathrm{Fam}_\Gamma(\sigma_\lambda,\varphi)$ *be the subspace of all deformable invariant distributions which are strongly supported on the limit set.*

Being deformable is a non-trivial condition. For example, if Γ has finite-covolume, then the boundary values of cusp forms are not deformable, while the boundary values of Eisenstein series are deformable. We also have examples of non-deformable invariant distributions in the case of convex cocompact Γ (see [3], [9], [5]).

Using *embedding* and *twisting* we can formulate a weaker condition. We first describe embedding. For a moment we write G_n for $Spin(1,n)$ or $SO(1,n)_0$, and we attach the subscript n to all related objects. For $n\geq m$ have inclusions $G_m\subset G_n$, $\partial X_m\subset\partial X_n$, $P_m\subset P_n$, $M_m\subset M_n$. Note that $A_m\cong A_n=:A$ and thus $\mathfrak{a}^*_{\mathbb{C}m}\cong\mathfrak{a}^*_{\mathbb{C}n}=:\mathfrak{a}^*_{\mathbb{C}}$ in a natural way.

Let $\sigma_n\in\hat M_n$, $\sigma_m\in\hat M_m$ and assume that there is an M_m-invariant inclusion $T:V_{\sigma_m}\to V_{\sigma_n}$. Then T^* induces a projection $V^n((\tilde\sigma_n)_{-\lambda-\rho^m+\rho^n})_{|\partial X^n}\to V^m((\tilde\sigma_m)_{-\lambda})$. Let

$$i^*_{n,\sigma_n,T}: C^\infty(\partial X^n, V^n((\tilde\sigma_n)_{-\lambda-\rho^m+\rho^n},\tilde\varphi))\to C^\infty(\partial X^m, V^m((\tilde\sigma_m)_{-\lambda},\tilde\varphi))$$

be the restriction of functions and

$$i_*^{n,\sigma_n,T}: C^{-\infty}(\partial X^m, V^m((\sigma_m)_\lambda,\varphi))\to C^{-\infty}(\partial X^n, V^n((\sigma_n)_{\lambda+\rho^m-\rho^n},\varphi))$$

be the dual map. Note that $i_*^{n,\sigma_n,T}$ is injective and G_m-equivariant. For fixed m and σ_m the various choices of n, σ_n and T are are called embedding data. We will often abbreviate $i_*^{n,\sigma_n,T}=i_*$.

Now we explain twisting. If (π, V_π) is a finite-dimensional representation of G, then there is an isomorphism of Γ-equivariant bundles $R : V(\theta\otimes\pi_{|P}, \varphi) \xrightarrow{\sim} V(\theta, \varphi\otimes\pi)$ given by $R([g, x\otimes y]\otimes z) := [g, x] \otimes z \otimes \pi(g)y$. Fix $\sigma \in \hat{M}$ and $\lambda \in \mathfrak{a}^*_\mathbb{C}$. Assume that we are given $\sigma' \in \hat{M}$, $\mu \in \mathfrak{a}^*_\mathbb{C}$, a finite-dimensional representation of (π, V_π) of G, and a P-equivariant inclusion $T : V_{\sigma_\lambda} \hookrightarrow V_{\sigma'_\mu} \otimes V_\pi$. Then we obtain a Γ-equivariant inclusion

$$i_*^{\sigma',\mu,\pi,T} : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^{-\infty}(\partial X, V(\sigma'_\mu \otimes \pi_{|P}, \varphi)) \stackrel{R}{\to} C^{-\infty}(\partial X, V(\sigma'_\mu, \varphi \otimes \pi)) \ .$$

Note that $i_*^{\sigma',\mu,\pi,T}$ is P_Γ-equivariant for any Γ-cuspidal parabolic subgroup P. For simplicity we will from now on assume that (π, V_π) is *irreducible*. The various choices of σ', μ, π and T are called twisting data. We often abbreviate $i_*^{\sigma',\mu,\pi,T} = i_*$.

We also allow iterated twisting and embedding, and we still denote the resulting inclusion by i_*. We say that $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ is *stably deformable* if $i_*\phi$ becomes deformable for suitable embedding or twisting data.

Definition 1.6 *Let* $\mathrm{Fam}^{st}_\Gamma(\sigma_\lambda, \varphi)$ *denote the subspace of stably deformable invariant distributions.*

Being stably deformable is a weaker condition than being deformable. E.g., if Γ has finite covolume, then the boundary values of cusp forms are not deformable, but generically they are stably deformable. In fact, for most $\lambda \in \mathfrak{a}^*_\mathbb{C}$ all invariant distributions are stably deformable. Let $I_\mathfrak{a} \subset \mathfrak{a}^*$ denote the lattice of half-integral characters spanned by $\frac{1}{2}\alpha$.

Theorem 1.7 *If* $\lambda \notin I_\mathfrak{a}$, *then* $\mathrm{Fam}^{st}_\Gamma(\sigma_\lambda, \varphi) = {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$.

A proof will appear in a future publication. One of the main steps is to show that res^Γ is regular (i.e. $res^\Gamma \phi_\mu$ is holomorphic for any germ at λ of a holomorphic family $(\phi_\mu)_\mu$ of Γ-invariant distributions) for $\lambda \notin I_\mathfrak{a}$ for any Γ which is elementary parabolic, i.e. $\Gamma = \Gamma_P$ for some Γ-cuspidal parabolic subgroup.

The following was shown in [5], proof of Cor. 6.12.

Theorem 1.8 *If* Γ *is convex cocompact, then*

$$\mathrm{Fam}^{st}_\Gamma(\sigma_\lambda, \varphi) = {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$$

holds for any $\lambda \in \mathfrak{a}^*_\mathbb{C}$.

If Γ is convex cocompact and not cocompact, then twisting alone suffices to make an invariant distribution deformable. We do not know any example of an invariant distribution which is not stably deformable.

Conjecture 1.9 *If Γ is geometrically finite, then*

$$\mathrm{Fam}^{st}_\Gamma(\sigma_\lambda,\varphi) = {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$$

for all σ, λ and φ.

Note that if $\phi \in \mathrm{Fam}^{st}_\Gamma(\sigma_\lambda,\varphi)$, then we can always find embedding and twisting data such that $i_*\phi \in \mathrm{Fam}_\Gamma(1_{\lambda'},\varphi')$. This will be employed later.

If $\phi \in \mathrm{Fam}^{st}_\Gamma(\sigma_\lambda,\varphi)$, then we say that it is *strongly supported on the limit set* if $i_*\phi$ is strongly supported on the limit set for some choice of embedding and twisting data with the property that $i_*\phi$ is deformable. This definition makes sense since being strongly supported on the limit set is stable under twisting and embedding by Lemma 2.13.

Definition 1.10 *We let* $\mathrm{Fam}^{st}_\Gamma(\Lambda_\Gamma,\sigma_\lambda,\varphi)$ *be the space of stably deformable invariant distribution vectors which are strongly supported on the limit set.*

An alternative way to sharpen the condition "supported on the limit set" is to require a "cusp form" condition. If $\lambda \not\in I_{\mathfrak{a}}$, then "cusp form" implies "strongly supported on the limit set" (Theorem 1.14).

Let P be a Γ-cuspidal parabolic subgroup. We normalize the Haar measure dx on the unimodular group P_Γ such that $\mathrm{vol}(\Gamma_P\backslash P_\Gamma)=1$. Note that $\pi^{\sigma_\lambda,\varphi}$ extends to P_Γ.

Definition 1.11 *If $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$, then we define its* constant term

$$\phi_P \in {}^{P_\Gamma} C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$$

with respect to P by

$$\phi_P := \int_{\Gamma_P\backslash P_\Gamma} \pi^{\sigma_\lambda,\varphi}(x)\phi\, dx\ .$$

Definition 1.12 *We call $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$ a* cusp form *if* $\mathrm{supp}(\phi) \subset \Lambda_\Gamma$ *and* $\phi_P = 0$ *for all Γ-cuspidal parabolic subgroups P.*

If Γ has finite covolume and $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda))$ is a cusp form, then the matrix coefficients $G \ni g \mapsto \langle \phi, \pi^{\tilde\sigma_{-\lambda}}(g) f\rangle$ for K-finite $f \in C^\infty(\partial X, V(\tilde\sigma_{-\lambda}))$ are cusp forms in the classical sense. If Γ is convex cocompact, then any $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$ with $\mathrm{supp}(\phi) \subset \Lambda_\Gamma$ is a cusp form for trivial reasons. If ϕ is a cusp form, then $i_*\phi$ is also a cusp form for any choice of embedding or twisting data.

Definition 1.13 *Let* $\mathrm{Cusp}_\Gamma(\sigma_\lambda,\varphi)$ *denote the space of all cusp forms in* ${}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$.

Theorem 1.14 *If $\lambda \notin I_{\mathfrak a}$, then* $\mathrm{Cusp}_\Gamma(\sigma_\lambda,\varphi) \subset \mathrm{Fam}^{st}_\Gamma(\Lambda_\Gamma,\sigma_\lambda,\varphi)$.

Note that the proof of this theorem uses Thm. 1.7 and the fact that res^Γ is regular outside $I_{\mathfrak a}$. Assuming these facts in the present paper we show that a cusp form is strongly supported on the limit set.

Since ∂X is a closed manifold there are natural Sobolev spaces $H^{p,r}(\partial X, V(\sigma_\lambda,\varphi))$, $p \in (1,\infty)$, $r \in \mathbb{R}$. For $r \in \mathbb{N}_0$ we let $H^{p,r}(\partial X, V(\sigma_\lambda,\varphi))$ be the space of those distributions in $C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$ which have L^p-integrable derivatives up to order r. For $r \in -\mathbb{N}$ we define $H^{p,r}(\partial X, V(\sigma_\lambda,\varphi))$ to be the dual of $H^{q,-r}(\partial X, V(\tilde\sigma_{-\lambda},\tilde\varphi))$, where $\frac{1}{p}+\frac{1}{q}=1$. For non-integral r we define $H^{p,r}(\partial X, V(\sigma_\lambda,\varphi))$ by complex interpolation (see Subsection 3.1). We also define

$$H^{p,<r}(\partial X, V(\sigma_\lambda,\varphi)) := \bigcap_{s<r} H^{p,s}(\partial X, V(\sigma_\lambda,\varphi)) \ .$$

Now we can state our main result.

Theorem 1.15 *Let $r_{p,\lambda}(\Gamma), r^0_{p,\lambda}(\Gamma) \in \mathbb{R}$ be determined by*

$$\begin{aligned} r^0_{p,\lambda}\alpha &:= \mathrm{Re}(\lambda) - \delta_\varphi - \rho + \frac{\rho-\delta_\Gamma}{p} \\ r_{p,\lambda}(\Gamma)\alpha &:= \min\left(r^0_{p,\lambda}(\Gamma)\alpha, \quad 2\mathrm{Re}(\lambda) + \frac{2\rho}{p} - \max_{cusps\,[P]_\Gamma}\left[2\delta_{\varphi_{|\Gamma_P}} + \alpha\,\mathrm{rank}([P]_\Gamma)\right]\right) \end{aligned}$$

(the maximum over the empty set is defined by convention as $-\infty$). Then

$$\begin{aligned} \mathrm{Fam}^{st}_\Gamma(\Lambda_\Gamma,\sigma_\lambda,\varphi) &\subset H^{p,<r_{p,\lambda}(\Gamma)}(\partial X, V(\sigma_\lambda,\varphi)) \\ \mathrm{Cusp}_\Gamma(\sigma_\lambda,\varphi) &\subset H^{p,<r^0_{p,\lambda}(\Gamma)}(\partial X, V(\sigma_\lambda,\varphi)) \qquad \text{if } \lambda \not\in I_{\mathfrak{a}} \ . \end{aligned}$$

Remarks :

1. If $\delta_\Gamma < \rho$, then it is clear that $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$ with $\mathrm{supp}(\phi) \subset \Lambda_\Gamma$ cannot belong to $H^{p,0}(\partial X, V(\sigma_\lambda,\varphi))$. Indeed, in this case the limit set Λ_Γ has trivial Lebesgue measure and cannot be the support of any non-trivial L^p-function. We conclude

   Corollary 1.16 *If $r_{1,\lambda}(\Gamma) > 0$ and $\delta_\Gamma < \rho$, then*
   $$\mathrm{Fam}^{st}_\Gamma(\Lambda_\Gamma,\sigma_\lambda,\varphi) = \{0\} \ .$$

   This is the vanishing result obtained in [4], Prop. 4.22, resp. [6]. Note that the Corollary does *not follow logically* from the present paper because it is *used* in the argument in order to deal with positive $r_{p,\lambda}(\Gamma)$.

2. Assume that Γ has finite covolume. In this case $\rho = \delta_\Gamma$. If we take $\sigma = 1$, $\varphi = 1$ and $\lambda = \rho$, then $r_{p,\lambda}(\Gamma) = 0$. However, the constant function on ∂X is a *smooth* Γ-invariant section of $V(1_\rho)$ which is strongly supported on the limit set.

3. Assume that Γ is convex cocompact, $\sigma = 1$, $\varphi = 1$, and $\lambda = \delta_\Gamma$. Then the *Patterson-Sullivan measure* generates $\mathrm{Fam}^{st}_\Gamma(\Lambda_\Gamma, 1_\lambda, 1)$. A measure belongs to $H^{p,<-\dim(\partial X)\frac{1}{q}}$ for all $p \in (1,\infty)$. In contrast, our estimate gives $r_{p,\lambda}(\Gamma)\alpha = -(\dim(X) - \dim_H(\Lambda))\frac{1}{q} > -\frac{1}{q}\dim(\partial X)\alpha$ except if Γ is elementary hyperbolic. Thus the Patterson-Sullivan measure is more regular than a general measure. It has the regularity of a smooth measure on Λ if the latter would be a smooth submanifold.

4. It follows from lower regularity bounds obtained by W. Schmid in special cases that our estimate is essentially optimal. We refer to Subsection 1.4.1 for more details.

5. Probably, the condition $\lambda \not\in I_{\mathfrak{a}}$ in the cusp form case is not necessary.

Let us now explain the modifications in the case that Γ has torsion. In this case there is a torsion-free subgroup $\Gamma' \subset \Gamma$ of finite index which is still geometrically finite. Moreover, we have $\Lambda_\Gamma = \Lambda_{\Gamma'}$ and $\delta_\Gamma = \delta_{\Gamma'}$. We define

$$\mathrm{Fam}^{st}_\Gamma(\Lambda_\Gamma,\sigma_\lambda,\varphi) := \mathrm{Fam}^{st}_{\Gamma'}(\Lambda_{\Gamma'},\sigma_\lambda,\varphi) \cap {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$$

(this definition does not depend on the choice of Γ'). Cusp forms are characterized as in the torsion-free case. Then Theorem 1.15 holds true for Γ with torsion elements.

1.2 Idea of the proof

Let us demonstrate the main ideas leading to the proof of Theorem 1.15 in the special case that $\delta_\varphi = 0$, $\sigma = 1$, $\mathrm{Re}(\lambda) < \min(-\delta_\Gamma, 0)$, $\lambda \notin I_{\mathfrak{a}}$, and that Γ has no cusps of full rank and is not cocompact.

Let $\phi \in \mathrm{Fam}_\Gamma(\Lambda_\Gamma, 1_\lambda, \varphi)$. We apply the Knapp-Stein intertwining operator (see Subsection 2.2) and obtain $\psi := \hat{J}^w_{1_\lambda,\varphi}\phi \in {}^\Gamma C^{-\infty}(\partial X, V(1_{-\lambda}, \varphi))$. The fact that ϕ is strongly supported on the limit set implies that ψ is smooth on Ω_Γ and has a controlled growth along the cusps. This will be formalized by saying that $\psi_{|\Omega_\Gamma}$ belongs to the space $B_\Gamma(1_{-\lambda}, \varphi)$ which will be introduced in Subsection 2.1. Using the convergence of the Poincaré series $\sum_{\gamma\in\Gamma} a_\gamma^{-(\delta_\Gamma+\rho+\epsilon)}$, $0 < \epsilon \in \mathfrak{a}^*$, the Γ-invariance of $\psi_{|\Omega_\Gamma}$, and the controlled growth along the cusps we show (Theorem 4.6) that $\psi \in H^{p,<r_{p,\lambda}(\Gamma)-2\frac{\mathrm{Re}(\lambda)}{\alpha}}$. Since $\lambda \notin I_{\mathfrak{a}}$ we recover ϕ from ψ by applying a multiple of $\hat{J}^{w^{-1}}_{1_{-\lambda},\varphi}$. This operator decreases the regularity by $-2\frac{\mathrm{Re}(\lambda)}{\alpha}$ (see Subsection 3.2) so that we obtain the desired result $\phi \in H^{p,<r_{p,\lambda}(\Gamma)}$. If ϕ is a cusp form, then we get a stronger estimate of ψ along the cusps leading to $\psi \in H^{p,<r^0_{p,\lambda}(\Gamma)-2\frac{\mathrm{Re}(\lambda)}{\alpha}}$ and $\phi \in H^{p,<r^0_{p,\lambda}(\Gamma)}$.

1.3 Specializations

In this subsection we discuss specializations of the main theorem. In our main theorem we quantify the regularity of distributions using the two-parameter family of spaces $H^{p,r}$. Let us indicate another way to measure the regularity of a distributions.

For $0 < s \in \mathbb{R}$, $s = [s] + s'$, $[s] \in \mathbb{N}_0$, $s' \in [0, 1)$ let $C^s(\partial X, V(\sigma_\lambda, \varphi))$ be the space of sections which have Hölder continuous (with exponent s') derivatives up to order $[s]$. Furthermore, let $C^{<s} := \bigcap_{t<s} C^t$. For negative s we define C^s (resp. $C^{<s}$) to be the space of distributions which can locally be written as N-fold derivatives of elements of C^{s+N} (resp. $C^{<s+N}$), where N is sufficiently large. For all $s \in \mathbb{R}$ and $p \in (1, \infty)$ have the embeddings $H^{p,<\frac{n-1}{p}+s} \hookrightarrow C^{<s}$. Since we deduce Hölder regularity from Sobolev regularity we cannot omit the $<$-sign. In some cases a direct approach to Hölder regularity may show C^s instead of $C^{<s}$-regularity (see Subsection 1.4.1).

1.3.1 Γ cocompact

Assume that Γ is cocompact. Then $\Lambda_\Gamma = \partial X$, $\delta_\Gamma = \rho$. Any invariant distribution is stably deformable by Theorem 1.8 and strongly supported on the limit set.

Theorem 1.17 *Assume that Γ is cocompact, and let*

$$r\alpha := \mathrm{Re}(\lambda) - \delta_\varphi - \rho \ .$$

Then

$${}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \subset H^{p,<r}(\partial X, V(\sigma_\lambda, \varphi))$$

for any $p \in (1, \infty)$.

Using the freedom to choose p arbitrary large we obtain the following corollary.

Corollary 1.18 *If Γ is cocompact, then*

$${}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda)) \subset C^{<-\frac{n-1}{2}+\frac{\mathrm{Re}(\lambda)}{\alpha}} \ .$$

1.3.2 Γ convex cocompact

Assume now that Γ is convex cocompact. If $0 \neq \phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ satisfies $\mathrm{supp}(\phi) \subset \Lambda_\Gamma$, then it is deformable by Theorem 1.8 and strongly supported on the limit set.

We can thus apply Theorem 1.15 to the space

$${}^\Gamma C^{-\infty}(\Lambda_\Gamma, V(\sigma_\lambda, \varphi)) := \{\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \,|\, \mathrm{supp}(\phi) \subset \Lambda_\Gamma\} \ .$$

Theorem 1.19 *Let*

$$r\alpha := \mathrm{Re}(\lambda) - \delta_\varphi - \rho + \frac{\rho - \delta_\Gamma}{p} \ .$$

Then

$${}^\Gamma C^{-\infty}(\Lambda_\Gamma, V(\sigma_\lambda, \varphi)) \subset H^{p,<r}(\partial X, V(\sigma_\lambda, \varphi)) \ .$$

Using the freedom to choose p arbitrary large we obtain

Corollary 1.20 *If Γ is convex cocompact, then*

$$^{\Gamma}C^{-\infty}(\Lambda_\Gamma, V(\sigma_\lambda,\varphi)) \subset C^{<-\frac{n-1}{2}+\frac{\mathrm{Re}(\lambda)-\delta_\varphi}{\alpha}} \ .$$

1.3.3 Γ has finite covolume

Now assume that Γ has finite covolume. Let $\phi \in {}^{\Gamma}C^{-\infty}(\partial X, V(\sigma_\lambda))$ be a cusp form and $\lambda \notin I_{\mathfrak{a}}$. Applying Theorem 1.15 and using that $\delta_\Gamma = \rho$ we obtain the following result.

Theorem 1.21 *Assume that Γ has finite covolume and $\lambda \notin I_{\mathfrak{a}}$, then*

$$\mathrm{Cusp}_\Gamma(\sigma_\lambda) \subset H^{p,<-\frac{n-1}{2}+\frac{\mathrm{Re}(\lambda)}{\alpha}}(\partial X, V(\sigma_\lambda))$$

for any $p \in (1,\infty)$.

Corollary 1.22 *If Γ has finite covolume and $\lambda \notin I_{\mathfrak{a}}$, then*

$$\mathrm{Cusp}_\Gamma(\sigma_\lambda) \subset C^{<-\frac{n-1}{2}+\frac{\mathrm{Re}(\lambda)}{\alpha}} \ .$$

More general we have the following estimate.

Theorem 1.23 *If Γ has finite covolume, then for $p \in (1,\infty)$ we have*

$$\mathrm{Fam}^{st}_\Gamma(\Lambda_\Gamma, \sigma_\lambda, \varphi) \subset H^{p,<-\frac{n-1}{q}+2\frac{\mathrm{Re}(\lambda)}{\alpha}}(\partial X, V(\sigma_\lambda)) \ .$$

Corollary 1.24 *If Γ has finite covolume, then*

$$\mathrm{Fam}^{st}_\Gamma(\Lambda_\Gamma, \sigma_\lambda, \varphi) \subset C^{<-(n-1)+2\frac{\mathrm{Re}(\lambda)}{\alpha}}(\partial X, V(\sigma_\lambda,\varphi)) \ .$$

1.3.4 Residues of Eisenstein series

For general geometrically finite Γ a source of interesting invariant distributions are boundary values of residues of Eisenstein series. In the language of geometric scattering theory [4] these are invariant distributions obtained from the singular part of the extension map ext^Γ. We refer to Subsection 2.2 for more details.

Definition 1.25 *Let* $\mathrm{Ext}^{sing}_\Gamma(\sigma_\lambda,\varphi)\subset {}^\Gamma C^{-\infty}(\partial X,V(\sigma_\lambda,\varphi))$ *be the subspace of invariant distributions generated by the singular part of* ext^Γ.

Essentially by definition the elements of $\mathrm{Ext}^{sing}_\Gamma(\sigma_\lambda,\varphi)$ belong to ${}^\Gamma C^{-\infty}(\partial X,V(\sigma_\lambda,\varphi))$ and are deformable. Moreover, the relation $res^\Gamma\circ ext^\Gamma=\mathrm{id}$ implies that the elements of $\mathrm{Ext}^{sing}_\Gamma(\sigma_\lambda,\varphi)$ are strongly supported on the limit set. We thus have $\mathrm{Ext}^{sing}_\Gamma(\sigma_\lambda,\varphi)\subset \mathrm{Fam}_\Gamma(\Lambda_\Gamma,\sigma_\lambda,\varphi)$. We show in Section 2.2 that these spaces are in fact equal.

Note that for $\mathrm{Re}(\lambda)>\delta_\Gamma+\delta_\varphi$ the extension ext^Γ is regular and thus $\mathrm{Ext}^{sing}_\Gamma(\sigma_\lambda,\varphi)=0$. Theorem 1.15 gives the following result.

Theorem 1.26 *We have*

$$\mathrm{Ext}^{sing}_\Gamma(\sigma_\lambda,\varphi)\subset H^{p,<r_{p,\lambda}(\Gamma)}(\partial X,V(\sigma_\lambda,\varphi))\ .$$

Choosing p arbitrary large we obtain the following consequence.

Corollary 1.27 *If* Γ *is geometrically finite, then*

$$\mathrm{Ext}^{sing}_\Gamma(\sigma_\lambda,\varphi)\subset C^{<r}\ ,$$

where

$$r:=\min\left(-\frac{n-1}{2}+\frac{\mathrm{Re}(\lambda)-\delta_\varphi}{\alpha},\ 2\frac{\mathrm{Re}(\lambda)}{\alpha}-\max_{cusps\,[P]_\Gamma}\left[2\frac{\delta_{\varphi_{|P_\Gamma}}}{\alpha}+\mathrm{rank}([P]_\Gamma)\right]\right)\ .$$

1.4 Related results

1.4.1 Results of W. Schmid

In [13] W. Schmid considers a finitely generated discrete subgroup Γ of $G=SL(2,\mathbb{R})$. Note that $G\cong Spin(1,2)$. If $\phi\in {}^\Gamma C^{-\infty}(\partial X,V(\sigma_\lambda))$, then it defines a G-equivariant map

$$c_\phi: C^\infty(\partial X,V(\tilde\sigma_{-\lambda}))\to C^\infty(\Gamma\backslash G)$$

by $c_\phi(f)(g):=\langle\phi,\pi^{\tilde\sigma_\lambda}(g)f\rangle$. Let us call ϕ *square integrable*, iff $c_\phi(f)\in L^2(\Gamma\backslash G)$ for all $f\in C^\infty(\partial X,V(\tilde\sigma_{-\lambda}))$. Schmid calls a square integrable ϕ (or rather the map c_ϕ) *cuspidal*, if $c_\phi(f)$ is a bounded function for all K-finite f, where K is some maximal compact subgroup of G.

Theorem 1.28 (Schmid) *Let $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda))$ be square integrable.*

1. *If ϕ is cuspidal and $\frac{\lambda}{\alpha} \notin \mathbb{Z} + \frac{1}{2}$, then ϕ is of Hölder regularity $C^{\frac{\mathrm{Re}(\lambda)-\rho}{\alpha}}$.*
2. *If $\frac{\lambda}{\alpha} \in \mathbb{Z} + \frac{1}{2}$, then ϕ is cuspidal and belongs to $C^{<\frac{\lambda-\rho}{\alpha}}$.*
3. *If ϕ is not cuspidal, then $\phi \in C^{<2(\lambda-\rho)}$ if $\lambda \in (0,\rho)$, and $\phi \in C^{<-1}$, if $\lambda \in (-\rho, 0)$.*

Note that in the two-dimensional case a finitely generated discrete group is geometrically finite, so the classes of discrete subgroups considered by Schmid and in the present paper coincide. On the one hand, being a square integrable invariant distribution is a strong restriction on ϕ. On the other hand there are square integrable distributions which are not strongly supported on the limit set. If ϕ is cuspidal and $\lambda \notin I_\mathfrak{a}$, or if ϕ is stably deformable, then our estimate applies to ϕ and reproduces Schmid's estimate. In fact, in the cuspidal case the result Schmid is slightly stronger than ours since he shows $\phi \in C^r$, while we show $\phi \in C^{<r}$ for the appropriate r.

In the case $\lambda \in (0,\rho)$ Schmid shows that his lower bounds of regularity are optimal in both, the cuspidal and the general case. This implies that our result is essentially optimal among estimates which can be stated using the same data and are compatible with embedding and twisting.

We thank W. Schmid for showing us a counterexample to an estimate claimed in a previous version of this paper.

1.4.2 Results of J. Lott

In [8] Lott considers discrete subgroups Γ of $SO(1,n)_0$. Let $\sigma^k := \Lambda^k \bar{\mathfrak{n}} \in \hat{M}$ and $\lambda_k := \rho - k\alpha$. Note that $V(\sigma^k_{\lambda_k}) = \Lambda^k T^* \partial X$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Then $\Lambda^k \mathfrak{p} \in \hat{K}$ is the minimal K-type of $\pi^{\tilde{\sigma}^k_{-\lambda_k}}$. Fix $0 \neq f_k \in C^\infty(\partial X, V(\tilde{\sigma}^k_{-\lambda_k}))(\Lambda^k \mathfrak{p})$. The main result of Lott is the following theorem.

Theorem 1.29 (Lott) *If $\phi \in C^{-\omega}(\partial X, V(\sigma^k_{\lambda_k}))$ is exact (as a differential form) and such that $c_\phi(f_k) \in C^\infty(G)$ is bounded, then $\phi \in H^{2,<-k}$.*

Among the variety of results proved in Lott's paper the following can be compared with those of the present paper.

Theorem 1.30 (Lott) *Let Γ be convex cocompact.*

1. *If $k \in [1, \frac{n-1}{2})$, and $\phi \in {}^\Gamma C^{-\infty}(\Lambda_\Gamma, V(\sigma^k_{\lambda_k}))$, then $\phi \in H^{2,<-k}$.*

2. *If Γ is not cocompact and $\phi \in {}^\Gamma C^{-\infty}(\Lambda_\Gamma, V(\sigma^1_{\lambda_1}))$, then $\phi \in H^{2,-1}$.*

This can directly be compared with our result, which gives $\phi \in H^{2,<-k+\frac{\rho-\delta_\Gamma}{2}}$. Thus if Γ is not cocompact, then our result improves Lott's estimate.

1.4.3 The result of J. Bernstein and A. Reznikov

In [2] J. Bernstein and A. Reznikov consider $G = SL(2,\mathbb{R}) \cong Spin(1,2)$, and a discrete subgroup of finite covolume. In Sec. 1.4 they "prove" the following result.

Theorem 1.31 (Bernstein-Reznikov) *If $\mathrm{Re}(\lambda) = 0$ and $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda))$ is square integrable, then $\phi \in (H^{1,1/2+\epsilon})^*$ for any $\epsilon > 0$.*

Our estimate applies if $\lambda \neq 0$, because in this case ϕ is a cusp form and $\lambda \notin I_{\mathfrak{a}}$. We show $\phi \in (H^{p,1/2+\epsilon})^*$ for any $\epsilon > 0$ and $p \in (1,\infty)$. This implies Bernstein-Reznikov's result.

2 Elements of geometric scattering theory

2.1 The space $B_\Gamma(\sigma_\lambda, \varphi)$

The spaces $B_\Gamma(\sigma_\lambda, \varphi)$ were introduced in [4]. In this subsection we recall some of their basic properties. First we consider the case that all cusps of Γ have smaller rank.

Recall that Γ acts properly discontinuously on the complement $\Omega_\Gamma = \partial X \setminus \Lambda_\Gamma$ of the limit set Λ_Γ with quotient B_Γ. Furthermore, we have a quotient bundle $V_{B_\Gamma}(\sigma_\lambda, \varphi) := \Gamma\backslash V(\sigma_\lambda, \varphi)_{|\Omega_\Gamma}$. The bundles $V_{B_\Gamma}(\sigma_\lambda, \varphi)$ depend holomorphically on $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. We consider the family of spaces $(C^\infty(B_\Gamma, V_{B_\Gamma}(\sigma_\lambda, \varphi)))_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}}$ as a (trivial) bundle of Fréchet spaces over $\mathfrak{a}^*_{\mathbb{C}}$.

For $\mathrm{Re}(\lambda) < -\delta_\Gamma - \delta_\varphi$ we can define the push-down map

$$\pi_*^\Gamma : C^\infty(\partial X, V(\sigma_\lambda,\varphi)) \to C^\infty(B_\Gamma, V_{B_\Gamma}(\sigma_\lambda,\varphi))$$

by $\pi_*^\Gamma(f) := \sum_{\gamma\in\Gamma} \pi^{\sigma_\lambda,\varphi}(\gamma) f_{|\Omega_\Gamma}$. Indeed, this sum converges and defines a smooth Γ-invariant section of $V(\sigma_\lambda,\varphi)_{|\Omega_\Gamma}$, and thus an element of $C^\infty(B_\Gamma, V_{B_\Gamma}(\sigma_\lambda,\varphi))$. One of the main results of [4] is that π_*^Γ has a meromorphic continuation to all of $\mathfrak{a}_\mathbb{C}^*$. If Γ is not convex-cocompact, then π_*^Γ is not surjective. Roughly speaking, the space $B_\Gamma(\sigma_\lambda,\varphi)$ is defined to be the range of π_*^Γ.

Definition 2.1 *If Γ has no cusps of full rank, then we define $(B_\Gamma(\sigma_\lambda,\varphi))_{\lambda\in\mathfrak{a}_\mathbb{C}^*}$ to be the minimal holomorphic subbundle of $(C^\infty(B_\Gamma, V_{B_\Gamma}(\sigma_\lambda,\varphi)))_{\lambda\in\mathfrak{a}_\mathbb{C}^*}$ over which π_*^Γ factors.*

This defines $B_\Gamma(\sigma_\lambda,\varphi)$ as a vector space. In fact, $f \in B_\Gamma(\sigma_\lambda,\varphi)$ iff there exists a meromorphic family $(F_\mu)_\mu$, $F_\mu \in C^\infty(\partial X, V(\sigma_\mu,\varphi))$, such that $(\pi_*^\Gamma F_\mu)_{\mu=\lambda} = f$.

As a topological vector space it will be equipped with a topology which is stronger than the topology induced from the embedding $B_\Gamma(\sigma_\lambda,\varphi) \to C^\infty(B_\Gamma, V_{B_\Gamma}(\sigma_\lambda,\varphi))$. With this stronger topology we cannot show that $(B_\Gamma(\sigma_\lambda,\varphi))_{\lambda\in\mathfrak{a}_\mathbb{C}^*}$ is a locally trivial bundle of Fréchet spaces. As explained in [4] we can consider it as a projective limit (see below) of locally trivial bundles of Fréchet spaces, and it therefore still makes sense to speak of holomorphic families $(f_\mu)_{\mu\in\mathfrak{a}_\mathbb{C}^*}$, $f_\mu \in B_\Gamma(\sigma_\mu,\varphi)$.

Note that $C_c^\infty(B_\Gamma, V_{B_\Gamma}(\sigma_\lambda,\varphi)) \subset B_\Gamma(\sigma_\lambda,\varphi)$, so that the elements of $B_\Gamma(\sigma_\lambda,\varphi)$ are distinguished among all smooth sections by their behaviour at the ends of B_Γ. In order to describe this behaviour in detail we first assume that $\Gamma \subset P$ for some Γ-cuspidal parabolic subgroup $P \subset G$. Then we have $\Omega_\Gamma = \partial X \setminus \infty_P$. We choose a maximal compact subgroup K and a Langlands decomposition $P = MAN$, $M \subset K$. Furthermore, let $w \in K$ be a representative of the non-trivial element of the Weyl group $N_K(M)/M$. Then we can parametrize $\Omega_\Gamma = NwP \subset G/P$. A function $f \in C^\infty(\Omega_\Gamma, V(\sigma_\lambda,\varphi))$ gives rise to a smooth function on N with values in $V_\sigma \otimes V_\varphi$ by

$$N \ni x \to f(xw) \in V_\sigma \otimes V_\varphi\ .$$

For $X \in \mathcal{U}(\mathfrak{n})$, $W \subset N$, and $\beta \in \mathfrak{a}^*$ we define

$$\begin{aligned} q^\Gamma_{W,X,\beta}(f) &:= \sup_{x\in W}\sup_{a\in A_+} a^{\beta-2(\mathrm{Re}(\lambda)-\rho^\Gamma)}|\varphi(a^{-1})f(x^aXw)| \qquad (2)\\ q_{W,X}(f) &= \sup_{x\in W}|f(xX)|\ . \end{aligned}$$

Note that in this context we extend φ to AP_Γ (which is possible since we assume that φ is admissible) such that the minimal A-weight is 0 and the highest A-weight is not greater than $2\delta_\varphi$.

The family of seminorms $q_{W,X}$ for all compact $W \subset N$ and $X \in \mathcal{U}(\mathfrak{n})$ defines the topology of $C^\infty(\Omega_\Gamma, V(\sigma_\lambda, \varphi))$.

Definition 2.2 *Let $S_{\Gamma,\beta,\kappa}(\sigma_\lambda, \varphi)$ be the closure of $C_c^\infty(B_\Gamma, V_{B_\Gamma}(\sigma_\lambda, \varphi))$ with respect to Banach space topology defined by the seminorms $q_{W,X}$, $\deg(X) \le \kappa$, $W \subset \Omega_\Gamma$ compact, and $q^\Gamma_{W,X,\beta}$, $\deg(X) \le \kappa$, $W \subset N \setminus N_\Gamma$ compact. Furthermore, we define the Fréchet spaces $S_{\Gamma,\beta}(\sigma_\lambda, \varphi) := \bigcap_{\kappa \in \mathfrak{a}^*} S_{\Gamma,\beta,\kappa}(\sigma_\lambda, \varphi)$ and $S_\Gamma(\sigma_\lambda, \varphi) := \bigcap_{\beta \in \mathfrak{a}^*} S_{\Gamma,\beta}(\sigma_\lambda, \varphi)$.*

The elements of $B_\Gamma(\sigma_\lambda, \varphi)$ will now be characterized by the property that they have a certain asymptotic expansion near ∞_P with remainder in $S_\Gamma(\sigma_\lambda, \varphi)$. In order to describe this asymptotic expansion we introduce the following space of homogeneous functions.

Definition 2.3 *For $\beta \in \mathfrak{a}^*$ we define $A_{P_\Gamma,\beta}(\sigma_\lambda, \varphi) \subset C^\infty(\Omega_\Gamma \setminus N_\Gamma w P, V(\sigma_\lambda, \varphi))$ to be the closed subspace of P_Γ-invariant sections satisfying*

$$\varphi(a)^{-1} f(x^a w) = a^{2(\lambda - \rho^\Gamma) - \beta} f(xw) \ .$$

In [4] we have shown that if $f \in B_\Gamma(\sigma_\lambda, \varphi)$, then it has an asymptotic expansion

$$\varphi(a^{-1}) f(x^a w) \overset{a \to \infty}{\sim} \sum_{k \in \mathbb{N}_0} a^{2(\lambda - \rho^\Gamma) - k\alpha} f_k(x) \ ,$$

where $f_k \in A_{P_\Gamma, k\alpha}(\sigma_\lambda, \varphi)$.

Definition 2.4 *We define $R_\Gamma(\sigma_\lambda, \varphi)^k \subset A_{P_\Gamma,k\alpha}(\sigma_\lambda, \varphi)$ to be the subspace generated by the f_k for all $f \in B_\Gamma(\sigma_\lambda, \varphi)$.*

The space $R_\Gamma(\sigma_\lambda, \varphi)^k$ is finite-dimensional. We further set $R_{\Gamma,k}(\sigma_\lambda, \varphi) := \bigoplus_{l \le k} R_\Gamma(\sigma_\lambda, \varphi)^l$, $B_{\Gamma,k}(\sigma_\lambda, \varphi) := S_{\Gamma,k\alpha}(\sigma_\lambda, \varphi) + B_\Gamma(\sigma_\lambda, \varphi)$, such that we have an exact sequence

$$0 \to S_{\Gamma,k\alpha}(\sigma_\lambda, \varphi) \to B_{\Gamma,k}(\sigma_\lambda, \varphi) \overset{AS}{\to} R_{\Gamma,k}(\sigma_\lambda, \varphi) \to 0 \ .$$

We can construct a split $L : R_{\Gamma,k}(\sigma_\lambda, \varphi) \to B_{\Gamma,k}(\sigma_\lambda, \varphi)$ of this exact sequence using a suitable P_Γ-invariant cut-off function. So we have $B_{\Gamma,k}(\sigma_\lambda, \varphi) = S_{\Gamma,k\alpha}(\sigma_\lambda, \varphi) \oplus L(R_{\Gamma,k}(\sigma_\lambda, \varphi))$, and

this decomposition defines a Fréchet space topology on $B_{\Gamma,k}(\sigma_\lambda,\varphi)$. Note that $B_\Gamma(\sigma_\lambda,\varphi) = \bigcap_{k\in\mathbb{N}} B_{\Gamma,k}(\sigma_\lambda,\varphi)$.

We need the following seminorms on $B_\Gamma(\sigma_\lambda,\varphi)$.

Definition 2.5 *If $W \subset N \setminus N_{\Gamma_P}$ is compact, $k \in \mathbb{N}_0$, and $X \in \mathcal{U}(\mathfrak{n})$ is homogeneous of degree* $\deg(X) \in \mathfrak{a}^*$, *then we define $p^\Gamma_{W,X}(f)$ by*

$$p^\Gamma_{W,X}(f) := \sup_{a\in A_+}\sup_{x\in W} a^{\deg(X)-2(\mathrm{Re}(\lambda)-\rho^\Gamma)}|\varphi(a^{-1})f(x^aXw)| \ .$$

Lemma 2.6 *$p^\Gamma_{W,X}$ is a continuous seminorm on $B_\Gamma(\sigma_\lambda,\varphi)$.*

Proof. Fix $k \in \mathbb{N}$ such that $k\alpha > \deg(X)$. Let $f \in B_{\Gamma,k}(\sigma_\lambda,\varphi)$. We define f_l such that $AS(f) = \oplus_{l\le k} f_l$. Furthermore, we set $g := f - LAS(f) \in S_{\Gamma,k\alpha}(\sigma_\lambda,\varphi)$. Note that f_l and g depend continuously on f. The restriction of $p^\Gamma_{W,X}$ to the space $S_{\Gamma,k\alpha}(\sigma_\lambda,\varphi)$ belongs to the family of seminorms which define the topology of $S_{\Gamma,k\alpha}(\sigma_\lambda,\varphi)$. So it remains to show that $A_{P_\Gamma,k\alpha}(\sigma_\lambda,\varphi) \ni f_l \to p^\Gamma_{W,X}(Lf_l)$ is continuous.

Let χ_Γ be the P_Γ-invariant cut-off function used to define L. Furthermore, let $\Delta(X) = \sum_j Y_j \otimes Z_j$ be the coproduct. Then we can write

$$\begin{aligned}
\varphi(a^{-1})L(f_l)(x^aXw) &= \varphi(a^{-1})\sum_j \chi(x^aY_jw)f_l(x^aZ_jw)\\
&= \varphi(a^{-1})\sum_j \chi(x^aY_jw)f_l((xZ_j^{a^{-1}})^a w)\\
&= \sum_j a^{2(\lambda-\rho^\Gamma)-\deg(Z_j)-l\alpha}\chi(x^aY_jw)f_l(xZ_jw)\\
&= a^{2(\lambda-\rho^\Gamma)-\deg(X)-l\alpha}\chi(x^aw)f_l(xXw)\\
&\qquad + \sum_{j,\deg(Y_j)>0} \chi(x^awY_j)f_l((xZ_j^{a^{-1}})^a w) \ .
\end{aligned}$$

The summands for $\deg(Y_j) > 0$ vanish for large $a \in A_+$ uniformly for $x \in W$. We therefore can majorize $p^\Gamma_{W,X}(Lf_l)$ by a continuous seminorm of $A_{P_\Gamma,l\alpha}(\sigma_\lambda,\varphi)$. □

We can now describe $B_\Gamma(\sigma_\lambda,\varphi)$ for a general geometrically finite torsion-free group Γ without cusps of full rank. If $P\subset G$ is a Γ-cuspidal parabolic subgroup, then there is a representative $B_P\subset B_\Gamma$ of an end of B_Γ which is isomorphic to a representative of the end of B_{Γ_P}. Let $\chi_P\in C^\infty(B_\Gamma)$ be supported on B_P such that it is identically one near infinity. If $f\in C^\infty(B_\Gamma,V_{B_\Gamma}(\sigma_\lambda,\varphi))$, then we can consider $\chi_P f$ as a section $T_P f\in C^\infty(B_{\Gamma_P},V_{B_{\Gamma_P}}(\sigma_\lambda,\varphi))$. Then $f\in B_\Gamma(\sigma_\lambda,\varphi)$ iff $T_Pf\in B_{\Gamma_P}(\sigma_\lambda,\varphi)$ for all Γ-cuspidal parabolic subgroups $P\subset G$.

Definition 2.7 *As a topological vector space $B_\Gamma(\sigma_\lambda,\varphi)$ is equipped with the smallest topology such that the inclusion $B_\Gamma(\sigma_\lambda,\varphi)\subset C^\infty(B_\Gamma,V_{B_\Gamma}(\sigma_\lambda,\varphi))$ and all the maps T_P are continuous.*

In a similar manner we define the topological vector spaces $B_{\Gamma,k}(\sigma_\lambda,\varphi)$ ($S_{\Gamma,\beta}(\sigma_\lambda,\varphi)$) to be the space of all $f\in C^\infty(B_\Gamma,V_{B_\Gamma}(\sigma_\lambda,\varphi))$ such that $T_Pf\in B_{\Gamma_P,k}(\sigma_\lambda,\varphi)$ ($T_Pf\in S_{\Gamma_P,\beta}(\sigma_\lambda,\varphi)$) for all Γ-cuspidal parabolic subgroups $P\subset G$. Furthermore, we set $R_{\Gamma,k}(\sigma_\lambda,\varphi):=\bigoplus_{P\in\tilde{\mathcal{P}}}R_{\Gamma_P,k}(\sigma_\lambda,\varphi)$, where $\tilde{\mathcal{P}}$ denotes a set of representatives of the Γ-conjugacy classes of Γ-cuspidal parabolic subgroups ($\tilde{\mathcal{P}}$ parametrizes the ends of B_Γ). Then we have a split exact sequence

$$0\to S_{\Gamma,k\alpha}(\sigma_\lambda,\varphi)\to B_{\Gamma,k}(\sigma_\lambda,\varphi)\stackrel{AS}{\to} R_{\Gamma,k}(\sigma_\lambda,\varphi)\to 0\ ,$$

and we can write

$$B_{\Gamma,k}(\sigma_\lambda,\varphi)=S_{\Gamma,k\alpha}(\sigma_\lambda,\varphi)+L(R_{\Gamma,k}(\sigma_\lambda,\varphi))\ . \tag{3}$$

We now describe the necessary modification if Γ has cusps of full rank. In this case the ends of B_Γ are parametrized by the set of Γ-conjugacy classes of Γ-cuspidal subgroups $\mathcal{P}^<$ corresponding to cusps of smaller rank. The same construction as above gives spaces $B_\Gamma(\sigma_\lambda,\varphi)_1$, $B_{\Gamma,k}(\sigma_\lambda,\varphi)_1$. The space $B_{\Gamma,k}(\sigma_\lambda,\varphi)$ will be the direct sum of $B_{\Gamma,k}(\sigma_\lambda,\varphi)_1$ and a finite-dimensional vector space $B_\Gamma(\sigma_\lambda,\varphi)_2:=\oplus_{P\in\tilde{\mathcal{P}}^{max}}R_{\Gamma_P}(\sigma_\lambda,\varphi)$, where $\tilde{\mathcal{P}}^{max}$ is a set of representatives of Γ-conjugacy classes Γ-cuspidal parabolic subgroups corresponding to cusps of full rank.

The space $R_{\Gamma_P}(\sigma_\lambda,\varphi)$ is the fibre of a trivial holomorphic vector bundle as follows. Let ${}^{\Gamma_P}\mathcal{E}_{\infty_P}(\sigma,\varphi)$ denote the sheaf of holomorphic families $f_\nu\in{}^{\Gamma_P}C^{-\infty}(\partial X,V(\sigma_\nu,\varphi))$ with $\mathrm{supp}(f)=\infty_P$. Since ${}^{\Gamma_P}\mathcal{E}_{\infty_P}(\sigma,\varphi)$ is torsion-free it is the space of sections of a unique holomorphic vector bundle ${}^{\Gamma_P}E_{\infty_P}(\sigma,\varphi)$ over $\mathfrak{a}^*_{\mathbb{C}}$. By ${}^{\Gamma_P}E_{\infty_P}(\sigma_\lambda,\varphi)$ we denote the fibre of ${}^{\Gamma_P}E_{\infty_P}(\sigma,\varphi)$ at $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$. Then we define $R_{\Gamma_P}(\sigma_\lambda,\varphi):={}^{\Gamma_P}E_{\infty_P}(\tilde\sigma_{-\lambda},\tilde\varphi)^*$.

The push-down π^Γ_* now decomposes as $(\pi^\Gamma_*)_1\oplus(\pi^\Gamma_*)_2$, where $(\pi^\Gamma_*)_1$ is the average as in the case without cusps of full rank. The second component further decomposes as $(\pi^\Gamma_*)_2=\oplus_{P\in\tilde{\mathcal{P}}^{max}}[\pi^\Gamma_*]_P$,

where $[\pi_*^\Gamma]_P$ has values in $R_{\Gamma_P}(\sigma_\lambda,\varphi)$. For $\mathrm{Re}(\lambda) < -\delta_\Gamma - \delta_\varphi$ it is defined by the condition

$$\langle \phi, [\pi_*^\Gamma]_P(f)\rangle = \sum_{[g]\in\Gamma/\Gamma_P} \langle \pi^{\tilde\sigma_{-\lambda},\tilde\varphi}(g)\phi, f\rangle$$

for all $\phi \in {}^{\Gamma_P}E_{\infty_P}(\tilde\sigma_{-\lambda},\tilde\varphi)$, and for general λ by meromorphic continuation.

The family of spaces $(B_{\Gamma,k}(\sigma_\lambda,\varphi))_{\lambda\in\mathfrak{a}^*_\mathbb{C}}$ forms a trivial holomorphic bundle of Fréchet spaces over $\mathfrak{a}^*_\mathbb{C}$. In the present paper we do not need an explicit trivialization, and we therefore omit its description, which can be found in [4].

2.2 Restriction, extension, and the scattering matrix

In the preceding subsection we have defined the Fréchet spaces $B_\Gamma(\sigma_\lambda,\varphi)$.

Definition 2.8 *We define $D_\Gamma(\sigma_\lambda,\varphi) := B_\Gamma(\tilde\sigma_{-\lambda},\tilde\varphi)^*$.*

Furthermore, we introduce

$$D_{\Gamma,k}(\sigma_\lambda,\varphi) := B_{\Gamma,k}(\tilde\sigma_{-\lambda},\tilde\varphi)^* \ .$$

The family $(D_{\Gamma,k}(\sigma_\lambda,\varphi))_{\lambda\in\mathfrak{a}^*_\mathbb{C}}$ is a trivial holomorphic bundle of dual Fréchet spaces. Moreover, we have $D_\Gamma(\sigma_\lambda,\varphi) = \lim\limits_{\overrightarrow{k}} D_{\Gamma,k}(\sigma_\lambda,\varphi)$. Therefore we can speak of holomorphic families $(\phi_\mu)_{\mu\in\mathfrak{a}^*_\mathbb{C}}$, $\phi_\mu \in D_\Gamma(\sigma_\mu,\varphi)$. Locally, ϕ_μ is a holomorphic family in $D_{\Gamma,k}(\sigma_\mu,\varphi)$ for some fixed k.

Definition 2.9 *We define the extension map $ext^\Gamma : D_\Gamma(\sigma_\lambda,\varphi) \to C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$ to be the adjoint of π_*^Γ.*

If $(\phi_\mu)_{\mu\in\mathfrak{a}^*_\mathbb{C}}$, $\phi_\mu \in D_\Gamma(\sigma_\mu,\varphi)$, is a meromorphic family, then $ext^\Gamma\phi_\mu \in C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$ is a meromorphic family of Γ-invariant distributions.

Definition 2.10 *By $\mathrm{Ext}_\Gamma(\sigma_\lambda,\varphi)$ we denote the subspace of ${}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$ consisting of evaluations at λ of germs families $(ext^\Gamma\phi_\mu)_\mu$, where $(\phi_\mu)_\mu$, $\phi_\mu \in D_\Gamma(\sigma_\mu,\varphi)$, is a germ of a meromorphic family. By $\mathrm{Ext}^0_\Gamma(\sigma_\lambda,\varphi) \subset \mathrm{Ext}_\Gamma(\sigma_\lambda,\varphi)$ we denote the subspace of those evaluations, where ϕ_λ is regular.*

In [4] we have shown that for generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we have

$$\mathrm{Ext}_\Gamma^0(\sigma_\lambda, \varphi) = \mathrm{Ext}_\Gamma(\sigma_\lambda, \varphi) = {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \ .$$

Definition 2.11 *If $(\psi_\mu)_\mu$, $\psi_\mu \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\mu, \varphi))$, is a germ of a meromorphic family at λ, then we define $(res^\Gamma(\psi_\mu))_\mu$, $res^\Gamma(\psi_\mu) \in D_\Gamma(\sigma_\mu, \varphi)$, to be the unique germ of a meromorphic meromorphic family such that $ext^\Gamma(res^\Gamma(\psi_\mu)) = \psi_\mu$.*

Note that if $\psi \in \mathrm{Ext}_\Gamma^0(\sigma_\lambda, \varphi)$, then we can define $res^\Gamma \psi \in D_\Gamma(\sigma_\lambda, \varphi)$.

Recall Definition 1.25 of $\mathrm{Ext}_\Gamma^{sing}(\sigma_\lambda, \varphi)$.

Lemma 2.12 *We have the equality*

$$\mathrm{Ext}_\Gamma^{sing}(\sigma_\lambda, \varphi) = \mathrm{Fam}_\Gamma(\Lambda_\Gamma, \sigma_\lambda, \varphi) \ .$$

Proof. The inclusion $\mathrm{Ext}_\Gamma^{sing}(\sigma_\lambda, \varphi) \subset \mathrm{Fam}_\Gamma(\Lambda_\Gamma, \sigma_\lambda, \varphi)$ follows from the identity $res^\Gamma \circ ext^\Gamma = \mathrm{id}$, which holds when both sides are applied to germs of meromorphic families. The inclusion $\mathrm{Fam}_\Gamma(\Lambda_\Gamma, \sigma_\lambda, \varphi) \subset \mathrm{Ext}_\Gamma^{sing}(\sigma_\lambda, \varphi)$ follows from the similar identity $ext^\Gamma \circ res^\Gamma = \mathrm{id}$, which also holds after application to germs of meromorphic families. □

Next we show that the condition "strongly supported on the limit set" is stable under twisting and embedding.

Lemma 2.13 *If $i_* : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^{-\infty}(\partial X', V(\sigma'_{\lambda'}, \varphi'))$ is induced by some twisting or embedding data, then*

$$i_*(\mathrm{Fam}_\Gamma(\Lambda_\Gamma, \sigma_\lambda, \varphi)) \subset \mathrm{Fam}_\Gamma(\Lambda_\Gamma, \sigma'_{\lambda'}, \varphi') \ .$$

Proof. We show $i_*(\mathrm{Ext}_\Gamma^{sing}(\sigma_\lambda, \varphi)) \subset \mathrm{Ext}_\Gamma^{sing}(\sigma'_{\lambda'}, \varphi')$ and then apply Lemma 2.12. But the latter inclusion follows immediately from the identity (see [4], Sec. 2.3) $i_* \circ ext^\Gamma = ext^\Gamma \circ i_*^\Gamma$, where the maps ext^Γ on the two sides of course act on different spaces, and $i_*^\Gamma : D_\Gamma(\sigma_\lambda, \varphi) \to D_\Gamma(\sigma'_{\lambda'}, \varphi')$.□

We can now define the scattering matrix. We fix a maximal compact subgroup $K \subset G$ and a parabolic subgroup $P \subset G$. Let $P = MAN$ be the associated Langlands decomposition and $w \in N_K(M)/M$ be a representative of the non-trivial element of the Weyl group. Then we have the meromorphic family of Knapp-Stein intertwining operators $\hat{J}^w_{\sigma_\lambda,\varphi} : C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi)) \to C^{-\infty}(\partial X, V(\sigma^w_{-\lambda},\varphi))$, where $\sigma^w(m) = \sigma(m^{w^{-1}})$. If $\mathrm{Re}(\lambda) < 0$ and $f \in C^\infty(\partial X, V(\sigma_\lambda,\varphi))$, then it is given by

$$\hat{J}^w_{\sigma_\lambda,\varphi}(f)(g) = \int_{\bar{N}} f(gw\bar{n})d\bar{n} \ .$$

Let $(\phi_\mu)_{\mu\in\mathfrak{a}^*_\mathbb{C}}$, $\phi_\mu \in D_\Gamma(\sigma_\mu,\varphi)$, be a germ of a meromorphic family at λ. Then $(J^w_{\sigma_\mu,\varphi}(ext^\Gamma\phi_\mu))_\mu$, $J^w_{\sigma_\mu,\varphi}(ext^\Gamma\phi_\mu) \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma^w_{-\mu},\varphi))$, is a germ of a meromorphic family.

Definition 2.14 *We define the germ of a meromorphic family $(\hat{S}^w_{\sigma_\mu,\varphi}(\phi_\mu))_\mu$ at λ, $\hat{S}^w_{\sigma_\mu,\varphi}(\phi_\mu) \in D_\Gamma(\sigma^w_{-\mu},\varphi)$, by $\hat{S}^w_{\sigma_\mu,\varphi}(\phi_\mu) := res^\Gamma(\hat{J}^w_{\sigma_\mu,\varphi}(ext^\Gamma\phi_\mu))$.*

Note that for generic λ we have a well-defined continuous map $\hat{S}^w_{\sigma_\lambda,\varphi} : D_\Gamma(\sigma_\lambda,\varphi) \to D_\Gamma(\sigma^w_{-\lambda},\varphi)$. The identity

$$ext^\Gamma \circ \hat{S}^w_{\sigma_\mu,\varphi} = \hat{J}^w_{\sigma_\mu,\varphi} \circ ext^\Gamma \ .$$

is an immediate consequence of the definitions. Strictly speaking, this equation holds after applying both sides to a germ of a meromorphic family.

There is a natural pairing $V_{B_\Gamma}(\sigma_\lambda,\varphi)\otimes V_{B_\Gamma}(\tilde{\sigma}_{-\lambda},\tilde{\varphi}) \to V_{B_\Gamma}(-\rho)$. Since $V_{B_\Gamma}(-\rho)$ is the bundle of densities on B_Γ its sections can be integrated. In [4] we have shown that if $\max_{P\in\mathcal{P}^<}(\delta_{\varphi_{|P_\Gamma}} - \rho^{\Gamma_P}) < 0$, then we have a non-degenerate paring $B_\Gamma(\sigma_\lambda,\varphi) \otimes B_\Gamma(\tilde{\sigma}_{-\lambda},\tilde{\varphi}) \to \mathbb{C}$ and thus an inclusion $B_\Gamma(\sigma_\lambda,\varphi) \hookrightarrow D_\Gamma(\sigma_\lambda,\varphi)$. Furthermore, we have shown the following result

Lemma 2.15 *If $(f_\mu)_\mu$ is a germ at λ of a meromorphic family, $f_\mu \in B_\Gamma(\sigma_\mu,\varphi)$, then $(\hat{S}^w_{\sigma_\mu,\varphi}(f_\mu))_\mu$ is a germ at λ of a meromorphic family, $\hat{S}^w_{\sigma_\mu,\varphi}(f_\mu) \in B_\Gamma(\sigma^w_{-\mu},\varphi)$.*

We now prove Theorem 1.14 assuming that res^Γ is regular outside $I_\mathfrak{a}$. Let $\phi \in \mathrm{Cusp}(\sigma_\lambda,\varphi)$. By Theorem 1.7 we know that ϕ is stably deformable. After suitable twisting and embedding we can assume that $\phi \in \mathrm{Cusp}(\sigma_\lambda,\varphi) \cap \mathrm{Fam}_\Gamma(\sigma_\lambda,\varphi)$ and there are no cusps of full rank. Let $(\phi_\mu)_\mu$ be a germ of a holomorphic family such that $\phi_\lambda = \phi$. Since $\mathrm{supp}(\phi) \subset \Lambda_\Gamma$ we conclude

that $\{res^\Gamma\phi_\mu\}_{|\mu=\lambda}=0$, where $\{\psi\}$ denotes the restriction of $\psi\in D_\Gamma(\sigma_\lambda,\varphi)$ to $S_\Gamma(\tilde\sigma_{-\lambda},\tilde\varphi)\subset B_\Gamma(\tilde\sigma_{-\lambda},\tilde\varphi)$ (here we use that fact that $C_c^\infty(B_\Gamma,V_{B_\Gamma}(\tilde\sigma_{-\lambda},\tilde\varphi)$ is dense in $S_\Gamma(\tilde\sigma_{-\lambda},\tilde\varphi)$). Using the decomposition [4], (22), we can write

$$(res^\Gamma\phi_\mu)_{\mu=\lambda}=\sum_{P\in\tilde{\mathcal{P}}}(T_P^*res^{\Gamma_P}\phi_\mu)_{\mu=\lambda}\ .$$

It suffices to show that $(T_P^*res^{\Gamma_P}\phi_\mu)_{\mu=\lambda}=0$ for all $P\in\tilde{\mathcal{P}}$. Let $\chi\in C_c^\infty(P_\Gamma)$ be such that $\sum_{\gamma\in\Gamma_P}\gamma^*\chi\equiv 1$. Then we can write

$$\phi_P:=\int_{P_\Gamma}\chi(x)\pi^{\sigma_\lambda,\varphi}(x)\phi\,dx\ .$$

Since we assume that $\lambda\not\in I_{\mathfrak{a}}$ the push-down $\pi_*^{\Gamma_P}$ is regular and surjective at λ. In fact, the poles of $\pi_*^{\Gamma_P}$ are the same as the poles of $\pi_*^{P_\Gamma}$ (see the definition below). The location of the latter is determined in [4], Lemma 3.14. Furthermore, regularity of res^Γ is equivalent to surjectivity of $\pi_*^{\Gamma_P}$. Thus in order to check that $T_P^*(res^\Gamma\phi_\mu)_{\mu=\lambda}=0$ it suffices to show that $\langle\chi_P res^{\Gamma_P}\phi_\mu,\pi_*^{\Gamma_P}f_{-\mu}\rangle_{\mu=\lambda}=0$ for all germs $(f_\mu)_\mu$ at $-\lambda$ of holomorphic families $f_\mu\in C^\infty(\partial X,V(\tilde\sigma_\mu,\tilde\varphi))$. In fact, since $\{\chi_P res^{\Gamma_P}\phi_\mu\}_{\mu=\lambda}=0$ and $B_{\Gamma_P}(\tilde\sigma_\mu,\tilde\varphi)$ is a $C_c^\infty(B_{\Gamma_P})$-module it suffices to show

$$\langle res^{\Gamma_P}\phi_\mu,\pi_*^{\Gamma_P}f_{-\mu}\rangle_{\mu=\lambda}=0$$

for all families with $\mathrm{supp}(\pi_*^{\Gamma_P}f_\mu)\subset\{\chi_P=1\}$. We now consider

$$\pi_*^{P_\Gamma}(f_\mu):=\int_{P_\Gamma}\pi^{\tilde\sigma_\mu,\tilde\varphi}(x)(f_\mu)_{|\Omega_{\Gamma_P}}dx\ .$$

In [4] we have shown that $\pi_*^{P_\Gamma}(f_\mu)\in B_{\Gamma_P}(\tilde\sigma_\mu,\tilde\varphi)$, and that $(\pi^{\Gamma_P}(f_\mu)-\pi_*^{P_\Gamma}(f_\mu))_\mu$ is a germ of a holomorphic family in $S_\Gamma(\tilde\sigma_\mu,\tilde\varphi)$. Since $\pi_*^{P_\Gamma}$ factorizes over $\pi_*^{\Gamma_P}$ we also have $\mathrm{supp}(\pi_*^{P_\Gamma}f_\mu)\subset\{\chi_P=1\}$. Because of $\{\chi_P res^{\Gamma_P}\phi_\mu\}_{\mu=\lambda}=0$ we get

$$\langle res^{\Gamma_P}\phi_\mu,\pi_*^{\Gamma_P}f_{-\mu}\rangle_{\mu=\lambda}=\langle res^{\Gamma_P}\phi_\mu,\pi_*^{P_\Gamma}f_{-\mu}\rangle_{\mu=\lambda}\ . \tag{4}$$

We compute

$$\begin{aligned}\langle res^{\Gamma_P}\phi_\mu,\pi_*^{P_\Gamma}f_{-\mu}\rangle_{\mu=\lambda} &= \langle res^{\Gamma_P}\phi_\mu,\pi_*^{\Gamma_P}\int_{P_\Gamma}\chi(x)\pi^{\sigma_{-\mu},\tilde\varphi}(x)f_{-\mu}dx\rangle_{\mu=\lambda}\\ &= \langle ext^{\Gamma_P}res^{\Gamma_P}\phi_\mu,\int_{P_\Gamma}\chi(x)\pi^{\sigma_{-\mu},\tilde\varphi}(x)f_{-\mu}dx\rangle_{\mu=\lambda}\\ &= \left(\int_{P_\Gamma}\chi(x)\langle\phi_\mu,\pi^{\sigma_{-\mu},\tilde\varphi}(x)f_{-\mu}\rangle dx\right)_{\mu=\lambda}\\ &= \langle(\phi_\mu)_P,f_{-\mu}\rangle_{\mu=\lambda}\\ &= 0\ .\end{aligned}$$

We now apply (4). $\square$

3 Some functional analytic preparations

3.1 Complex interpolation

For $p \in (1,\infty)$ and $r \in \mathbb{N}_0$ the space $H^{p,r}(\partial X, V(\sigma_\lambda,\varphi))$ is the usual Sobolev space of sections of $V(\sigma_\lambda,\varphi)$ which have distributional derivatives in L^p up to order r. This space is well-defined, but there is no prefered norm. There are several slightly different ways to define Sobolev spaces of non-integral order. For our purpose a natural choice is complex interpolation. If $r_0, r_1 \in \mathbb{N}_0$, $\theta \in (0,1)$, and $r := r_0 + \theta(r_1 - r_0)$, then we define (see [11], IX.4, [1], Ch. 6)

$$H^{p,r}(\partial X, V(\sigma_\lambda,\varphi)) := [H^{p,r_0}(\partial X, V(\sigma_\lambda,\varphi)), H^{p,r_1}(\partial X, V(\sigma_\lambda,\varphi))]_\theta \ .$$

If $r \in \mathbb{N}_0$, then this definition coincides with the former. We extend the scale of Sobolev spaces to negative orders by duality

$$H^{p,-r}(\partial X, V(\sigma_\lambda,\varphi)) := H^{q,r}(\partial X, V(\tilde\sigma_{-\lambda},\tilde\varphi))^* \ ,$$

where $r \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for $r_0, r, r_1 \in \mathbb{R}$ with $r_0 < r < r_1$, $r = r_0 + \theta(r_1 - r_0)$ and $p_0, p, p_1 \in (1,\infty)$, $\frac{1}{p} = \frac{1}{p_0} + \theta(\frac{1}{p_1} - \frac{1}{p_0})$, we have

$$H^{p,r}(\partial X, V(\sigma_\lambda,\varphi)) := [H^{p_0,r_0}(\partial X, V(\sigma_\lambda,\varphi)), H^{p_1,r_1}(\partial X, V(\sigma_\lambda,\varphi))]_\theta \ . \tag{5}$$

We further will consider the Fréchet spaces

$$H^{p,<r}(\partial X, V((\sigma_\lambda,\varphi)) := \bigcap_{s<r} H^{p,s}(\partial X, V((\sigma_\lambda,\varphi)) \ .$$

For the convenience of the reader we recall the definition of the interpolation space (5). The pair of spaces

$$(H^{p_0,r_0}(\partial X, V(\sigma_\lambda,\varphi)), H^{p_1,r_1}(\partial X, V(\sigma_\lambda,\varphi)))$$

considered as subspaces of $C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$ forms a couple in the sense of [1], Sec. 2.3. There is a natural Banach norm on the sum $\Sigma := H^{p_0,r_0}(\partial X, V(\sigma_\lambda,\varphi)) + H^{p_1,r_1}(\partial X, V(\sigma_\lambda,\varphi))$. Let $\mathcal{F}$ be the space of all bounded, continuous functions f on the strip $\{\mathrm{Re}(z) \in [0,1]\}$ with

values in Σ which are holomorphic on the interior of this strip, such that for $t \in \mathbb{R}$ we have that $f(\mathrm{i}t) \in H^{p_0,r_0}(\partial X, V(\sigma_\lambda,\varphi))$ and $f(\mathrm{i}t+1) \in H^{p_1,r_1}(\partial X, V(\sigma_\lambda,\varphi))$ are continuous and

$$\sup_{t\in\mathbb{R}} \|f(\mathrm{i}t)\|_{H^{p_0,r_0}} < \infty \ , \quad \sup_{t\in\mathbb{R}} \|f(\mathrm{i}t+1)\|_{H^{p_1,r_1}} < \infty \ .$$

Then $\sup_{t\in\mathbb{R}}\{\|f(\mathrm{i}t)\|_{H^{p_0,r_0}}, \|f(\mathrm{i}t+1)\|_{H^{p_1,r_1}}\}$ is a norm making $\mathcal{F}$ into a Banach space. Let $\mathcal{F}_\theta$ be the closed subspace of all $f \in \mathcal{F}$ with $f(\theta) = 0$. Then

$$[H^{p_0,r_0}(\partial X, V(\sigma_\lambda,\varphi)), H^{p_1,r_1}(\partial X, V(\sigma_\lambda,\varphi))]_\theta := \mathcal{F}/\mathcal{F}_\theta \ .$$

We employ complex interpolation in the following way. We will show for a fixed distribution $f \in C^{-\infty}(\partial X, V(\sigma_\lambda,\varphi))$ that $f \in H^{p_0,r_0} \cap H^{p_1,r_1}$. If $\theta \in (0,1)$, $r = r_0 + \theta(r_1 - r_0)$ and $\frac{1}{p} = \frac{1}{p_0} + \theta(\frac{1}{p_1} - \frac{1}{p_0})$, then we conclude that $f \in H^{p,r}$. In order to see this we consider f as a constant function from $\{\mathrm{Re}(z) \in [0,1]\}$ to Σ and apply the interpolation result (5) above.

3.2 Regularity of intertwining operators

In this subsection we consider the Sobolev spaces $H^{p,r} := H^{p,r}(\mathbb{R}^n)$ for $p \in (1,\infty)$ and $r \in \mathbb{R}$ and the regularity properties of the operator $A_s : C_c^\infty(\mathbb{R}^n) \to C^{-\infty}(\mathbb{R}^n)$ which is given by the convolution kernel $a_s(x) := \|x\|^{-n-2s}$ for $s \in \mathbb{C}$, where $\|x\|^2 := \sum_{i=1}^n x_i^2$. To be precise, a_s is a regular distribution for $\mathrm{Re}(s) < 0$, and it is defined by meromorphic continuation for all $s \in \mathbb{C}$. Let $\Delta := \sum_{i=1}^n (\frac{\partial}{\partial x_i})^2$ be the Laplace operator. Then we have $\Delta\|x\|^{-n-2s} = (2s+n)(2s+2)\|x\|^{-n-2-2s}$. Iterating the equation

$$A_{s+1} = \frac{1}{(2s+n)(2s+2)} \Delta A_s \ . \tag{6}$$

we obtain the required meromorphic continuation.

For $\mathrm{Re}(s) = 0$, $s \neq 0$ the operator A_s is a singular integral operator. It is well-known that $A_s \in B(H^{p,r})$ for $p \in (1,\infty)$, $r \in \mathbb{R}$. If $\mathrm{Re}(s) \in \mathbb{N}_0$, then we can study the mapping properties of A_s by reduction to the case $\mathrm{Re}(s) = 0$ using (6). If $\mathrm{Re}(s) \geq 0$ is not integral, then we can reduce to the case $\mathrm{Re}(s) \in (-1,0)$. In this case A_s is given by convolution by a regular distribution. But A_s does not act nicely on $H^{p,r}$ since this distribution grows at infinity. So we must restrict the domain and extend the range.

Let $H_c^{p,r} \subset H^{p,r}$ be the subspace of all elements of compact support, and let $H_{loc}^{p,r} \subset C^{-\infty}(\mathbb{R}^n)$ be the space of all distributions f such that $\chi f \in H^{p,r}$ for all $\chi \in C_c^\infty(\mathbb{R}^n)$. We define $H_{loc}^{p,<r} := \bigcap_{s<r} H_{loc}^{p,s}$.

If $m \in \mathbb{N}_0$, then let $A_m = (s-m)^{-1}B_{m,-1} + B_{m,0} + \ldots$ be the Laurent expansion of A_s at $s = m$. Our main result is the following theorem.

Theorem 3.1 *If* $\mathrm{Re}(s_0) \geq 0$ *and* $s_0 \notin \mathbb{N}_0$*, then we have*

$$A_s : H_c^{p,r} \to H_{loc}^{p,r-2\mathrm{Re}(s_0)} \ .$$

For $m \in \mathbb{N}_0$ *we have*

$$\begin{aligned} B_{m,-1} &: H^{p,r} \to H^{p,r-2m} \ , \\ B_{m,k} &: H_c^{p,r} \to H_{loc}^{p,<r-2m} , \quad k \in \mathbb{N}_0 \ . \end{aligned}$$

Proof. The space $H_c^{p,r}$ is a limit of Banach spaces, while $H_{loc}^{p,r}$ is a Fréchet space in a natural way. For each compact $K \subset \mathbb{R}^n$ let $H_K^{p,r} := \{f \in H^{p,r} \mid \mathrm{supp}(f) \subset K\}$. Then $H_K^{p,r}$ is a closed subspace of $H^{p,r}$, and $H_c^{p,r} = \lim_{\overrightarrow{K}} H^{p,r}$. For each $\chi \in C_c^\infty(\mathbb{R}^n)$ we define the seminorm $q_{\chi,p,r}(f) := \|\chi f\|_{H^{p,r}}$ on $H_{loc}^{p,r}$. The family of these seminorms defines the topology of $H_{loc}^{p,r}$. If $A : H_c^{p,r} \to H_{loc}^{p,s}$ is continuous, then for each K and χ, we define

$$q(A)_{K,\chi,p,r,s} := \sup_{f \in H_K^{p,r}, \|f\|_{H^{p,r}}=1} q_{\chi,p,s}(Af) \ .$$

First we assume that $\mathrm{Re}(s) \in [-1, 0]$. In order to fix the pole at $s = 0$ we set $\hat{A} := sA_s$.

Proposition 3.2 *For each compact subset* $K \subset \mathbb{R}^n$ *and* $\chi \in C_c^\infty(\mathbb{R}^n)$ *there exists* $C < \infty$ *such that for all* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) \in [-1, 0]$ *we have*

$$q_{K,\chi,p,0,0}(\hat{A}_s) \leq C(1 + |s|)^3 \ .$$

Proof. Fix a compact $K \subset \mathbb{R}^n$ and $\chi \in C_c^\infty(\mathbb{R}^n)$. Let

$$R := \sup_{x \in K, y \in \mathrm{supp}(\chi)} \mathrm{dist}(x, y) \ . \tag{7}$$

Let $\kappa \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $\kappa(x) = 1$ for $\|x\| \leq R$ and $\kappa(x) = 0$ for $\|x\| \geq 2R$. We define $a_s^\kappa := \kappa a_s$ and let A_s^κ be the convolution operator defined by a_s^κ. If $\mathrm{supp}(f) \subset K$, then we have $\chi A_s f = \chi A_s^\kappa(f)$. It suffices to show

$$\|sA_s^\kappa\|_{B(L^p)} \leq C(1 + |s|)^3 \ ,$$

where C is independent of $s \in \{\mathrm{Re}(s) \in [-1,0]\}$. If $\mathrm{Re}(s) < 0$, then $a_s^\kappa \in L^1(\mathbb{R}^n)$, so that by Yang's inequality

$$\|A_s^\kappa\|_{B(L^p)} \leq \|a_s^\kappa\|_{L^1} \ .$$

There is a constant $C < \infty$ such that $\|a_s^\kappa\|_{L^1} \leq C$ uniformly for $\{\mathrm{Re}(s) \in [-1,-1/8]\}$, and hence

$$\|A_s^\kappa\|_{B(L^p)} \leq C \ . \tag{8}$$

We cannot use this estimate close to the imaginary axis, since in that region $\|a_s^\kappa\|_{L^1}$ explodes like $\mathrm{Re}(s)^{-1}$, and we need a uniform estimate. The point is that we must take into account the oscillating behaviour of $a_s^\kappa(x)$ near $x = 0$. Then the divergence above only occurs at $s = 0$, and this is fixed because we consider $\hat{A}_s^\kappa = sA_s^\kappa$.

Lemma 3.3 *There exists $C < \infty$ such that on* $\{\mathrm{Re}(s) \in [-1/4,0]\}$

$$\|\hat{A}_s^\kappa\|_{B(L^2)} \leq C(1+|s|) \ .$$

Proof. We first estimate $\|A_s^\kappa\|_{B(L^2)}$ using the Fourier transform. The Fourier transform $\mathcal{F}(a_s^\kappa)$ is the convolution of $\mathcal{F}(a_s)$ and $\mathcal{F}(\kappa) \in S(\mathbb{R}^n)$. We have

$$\mathcal{F}(a_s)(\xi) = c\, 2^{-2s} \frac{\Gamma(-s)}{\Gamma(\frac{1}{2}n+s)} \|\xi\|^{2s} \ ,$$

where c is independent of s. Note that (see [12], 6.328)

$$\lim_{|y|\to\infty} |\Gamma(x+\mathrm{i}y)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi} \ ,$$

locally uniformly for $x \in \mathbb{R}$. Using this property of the Γ-function we see that there is a constant $C < \infty$ such that for $\mathrm{Re}(s) \in [-1/4,0]$ we have $|s\frac{\Gamma(-s)}{\Gamma(\frac{1}{2}n+s)}| \leq C(1+\mathrm{Im}(s))^{-2\mathrm{Re}(s)-n/2+1}$. Let $\chi_1 \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $\chi_1(\xi) = 1$ for $\|\xi\| \leq 1$. Then uniformly on $\{\mathrm{Re}(s) \in [-1/4,0]\}$ we have $\|(1-\chi_1)\mathcal{F}(\hat{a}_s)\|_{L^\infty} \leq C(1+|s|)$. Furthermore, on $\{\mathrm{Re}(s) \in [-1/4,0]\}$ we obtain a uniform estimate $\|\chi_1\mathcal{F}(\hat{a}_s)\|_{L^1} \leq C(1+|s|)$. Writing $\mathcal{F}(\hat{a}_s^\kappa) = \mathcal{F}(\kappa) * \chi_1\mathcal{F}(\hat{a}_s) + \mathcal{F}(\kappa) * (1-\chi_1)\mathcal{F}(\hat{a}_s)$ we can estimate

$$\|\mathcal{F}(\hat{a}_s^\kappa)\|_{L^\infty} \leq \|\mathcal{F}(\kappa)\|_{L^\infty}\|\chi_1\mathcal{F}(\hat{a}_s)\|_{L^1} + \|\mathcal{F}(\kappa)\|_{L^1}\|(1-\chi_1)\mathcal{F}(\hat{a}_s)\|_{L^\infty} \leq C(1+|s|)$$

uniformly on $\{\mathrm{Re}(s) \in [1/4,0]\}$. This gives

$$\|\hat{A}_s^\kappa\|_{B(L^2)} \leq C(1+|s|)$$

uniformly on $\{\mathrm{Re}(s) \in [1/4,0]\}$. $\square$

In order to extend the assertion of Lemma 3.3 from L^2 to L^p we need the following estimate.

Lemma 3.4 *There exists a constant $C<\infty$ such that if $\{\mathrm{Re}(s)\in[-1/4,0]\}$, then*

$$\int_{\mathrm{dist}(y,z)\geq 4\mathrm{dist}(x,z)} |\hat{a}_s^\kappa(y-x)-\hat{a}_s^\kappa(y-z)|dy \leq C(1+|s|^3) \tag{9}$$

for all $x,z\in\mathbb{R}^n$.

Proof. On the domain of integration we have

$$\frac{1}{2}\mathrm{dist}(z,y)\leq \mathrm{dist}(x,y)\leq 2\mathrm{dist}(z,y)\ .$$

Recall the definition (7) of R. We decompose the domain of integration in (9) into the regions $\{\mathrm{dist}(z,y)\leq R/2\}$, $\{R/2\leq\mathrm{dist}(z,y)\leq 2R\}$, $\{\mathrm{dist}(z,y)\geq 2R\}$, and let I_1,I_2,I_3 denote the corresponding integrals. Since the integrand vanishes on the third region we have $I_3=0$. Next we consider I_1. In this region we can replace $\hat{a}_s^\kappa$ by $\hat{a}_s$. For $0\neq x\in\mathbb{R}^n$ let $x^0:=\frac{x}{\|x\|}$. We compute

$$\begin{aligned}
&|\hat{a}_s(y-z)-\hat{a}_s(y-x)| \qquad (10)\\
&= |s|\left|\|y-z\|^{-n-2s}-\|y-x\|^{-n-2s}\right|\\
&= |s|\|y-z\|^{-n-2\mathrm{Re}(s)}\left|1-\frac{\|y-z+z-x\|^{-n-2s}}{\|y-z\|^{-n-2s}}\right|\\
&= |s|\|y-z\|^{-n-2\mathrm{Re}(s)}\left|1-\|(y-z)^0+\frac{z-x}{\|y-z\|}\|^{-n-2s}\right|\\
&= |s|\|y-z\|^{-n-2\mathrm{Re}(s)}\left|1-\left(1+2\left\langle (y-z)^0,\frac{z-x}{\|y-z\|}\right\rangle+\frac{\|z-x\|^2}{\|y-z\|^2}\right)^{\frac{-n-2s}{2}}\right|\ . \qquad (11)
\end{aligned}$$

Note that $b:=2\langle (y-z)^0,\frac{z-x}{\|y-z\|}\rangle+\frac{\|z-x\|^2}{\|y-z\|^2}$ satisfies $|b|\leq 1/2+1/16<3/4$. For $b\in\mathbb{R}$, $|b|<3/4$ we write $(1+b)^{-n-2s}=\exp((-n-2s)\log(1+b))$. Furthermore, $\log(1+b)=b(1+r_1)$, where $\sup_{|b|\leq 3/4}|r_1|<\infty$. Then we expand $\exp((-n-2s)b(1+r_1))=1+(-n-2s)b(1+r_1)+br_2$, where $|r_2|\leq\frac{1}{2}|b|[(-n-2s)(1+r_1)]^2\sup_{\xi\in[0,1]}|\exp(\xi(-n-2s)b(1+r_1))|$. We conclude that $|r_2|<C(1+|s|)^2$, where C is independent of s and b. We further get $(1-(1+b)^{-n-2s})=(-n-2s)b+br_3$, where $|r_3|\leq C(1+|s|)^2$ uniformly in s. Inserting this estimate into (11) we obtain

$$|\hat{a}_s(y,z)-\hat{a}_s(y,x)|$$

$$\begin{aligned}
&= |s|\|y-z\|^{-n-2\mathrm{Re}(s)}\left|(2\left\langle (y-z)^0, \frac{z-x}{\|y-z\|}\right\rangle + \frac{\|z-x\|^2}{\|y-z\|^2})((-n-2s)+r_3)\right| \\
&= |s|\|y-z\|^{-n-2\mathrm{Re}(s)}\frac{\|z-x\|}{\|y-z\|}\left|(2\left\langle (y-z)^0, (z-x)^0\right\rangle + \frac{\|z-x\|}{\|y-z\|})((-n-2s)+r_3)\right| \\
&\leq (1+|s|)^3 C\|y-z\|^{-n}\frac{\|z-x\|}{\|y-z\|} \ .
\end{aligned}$$

We now estimate

$$\begin{aligned}
I_1 &\leq (1+|s|)^3 C\|z-x\| \int_{R/2\geq \mathrm{dist}(y,z)\geq 4\mathrm{dist}(x,z)} \|y-z\|^{-n-1} dy \\
&\leq (1+|s|)^3 C' \ ,
\end{aligned}$$

where C' can be chosen uniformly for $\mathrm{Re}(s) \in [-1/4, 0]$. Finally, on the compact region $\{R/2 \leq \mathrm{dist}(z,y) \leq 2R\}$ and for $\mathrm{Re}(s) \in [-1/4, 0]$ the integrand is uniformly bounded with respect to x, y, z. Thus

$$I_2 \leq C$$

uniformly on $\{\mathrm{Re}(s) \in [-1/4, 0]\}$. □

Lemma 3.5 *For $p \in (1, \infty)$ there exists $C < \infty$ such that on $\{\mathrm{Re}(s) \in [-1/4, 0)\}$*

$$\|\hat{A}_s^\kappa\|_{B(L^p)} \leq C(1+|s|)^3 \ .$$

Proof. We follow the argument of [7], Thm. III.2.4. There the following fact was shown. Let K be an integral operator given by an integral kernel $k(x,y)$. Assume that K extends to a bounded operator on L^2 and $\|K\|_{B(L^2)} \leq C_1$. Furthermore, assume that there are constants C_2, C_3 such that for all $z, x \in \mathbb{R}^n$

$$\int_{\mathrm{dist}(y,z)\geq C_2 \mathrm{dist}(x,z)} |k(y,x) - k(y,z)| dy \leq C_3 \ .$$

Then K extends to a bounded operator on L^p, $p \in (1, 2]$, and $\|K\|_{B(L^p)} \leq C(p, n, C_2)(1 + C_1 + C_3)$.

For $\mathrm{Re}(s) < 0$ the operator $\hat{A}_s^\kappa$ is given by an integral kernel. We set $C_2 = 4$. Then we obtain the assertion of the lemma by combining Lemma 3.4 (which estimates C_3) and Lemma 3.3 (which provides an estimate for C_1). For $p \in [2, \infty)$ we argue by duality using $(\hat{A}_s^\kappa)^* = \hat{A}_{\bar{s}}^\kappa$. □

We now finish the proof of Proposition 3.2. We must discuss the case $\mathrm{Re}(s) = 0$. Let $f \in S(\mathbb{R}^n)$. Then we know that in $S(\mathbb{R}^n)$ and thus in $L^p(\mathbb{R}^n)$

$$\lim_{\epsilon \downarrow 0} \hat{A}^\kappa_{s-\epsilon} f = \hat{A}^\kappa_s f \ .$$

Since we have the bound $\|\hat{A}^\kappa_{s-\epsilon}\|_{B(L^p)} \leq C(1+|s|)^3$ uniformly for $\epsilon \in (0,1)$ we conclude that $\|\hat{A}^\kappa_s\|_{B(L^p)} \leq C(1+|s|)^3$ uniformly if $\mathrm{Re}(s) \in [-1,0]$. □

Lemma 3.6 *Let $r \in \mathbb{R}$. For each compact subset $K \subset \mathbb{R}^n$ and $\chi \in C_c^\infty(\mathbb{R}^n)$ there exists $C < \infty$ such that for all $s \in \mathbb{C}$ with* $\mathrm{Re}(s) \in [-1,0]$ *we have*

$$q_{K,\chi,p,r,r}(\hat{A}_s) \leq C(1+|s|)^3 \ .$$

Proof. It again suffices to show that

$$\|\hat{A}^\kappa_s\|_{B(H^{p,r})} \leq C(1+|s|)^3 \ .$$

Let J_r be the operator on $S'(\mathbb{R}^n)$ given by $\mathcal{F}(J_r)f(\xi) = (1+|\xi|^2)^{r/2}\mathcal{F}(f)(\xi)$. Then it is well-known that $H^{p,r} = \{f \in S'(\mathbb{R}^n) \mid J_r f \in L^p\}$, and $\|f\|_{H^{p,r}} := \|J_r f\|_{L^p}$ is a norm on the Banach space $H^{p,r}$. Since $\hat{A}^\kappa_s$ commutes with translation, it also commutes with J_r (when restricted to the Schwartz space). So we can write $\hat{A}^\kappa_s = J_{-r}\hat{A}^\kappa_s J_r$. The right hand side obviously acts on $H^{p,r}$, and we have the estimate

$$\|\hat{A}^\kappa_s\|_{B(H^{p,r})} \leq \|\hat{A}^\kappa_s\|_{B(L^p)} \leq C(1+|s|)^3 \ .$$

This proves the lemma. □

We now consider the case that $\mathrm{Re}(s) \in [m-1,m]$, $m \in \mathbb{N}$. In order to fix the poles of A_s at $s = m-1, m$ we consider $\tilde{A}_{m,s} := s(s+1)A_{s+m}$, $\mathrm{Re}(s) \in [-1,0]$.

Lemma 3.7 *Let $r \in \mathbb{R}$. For each compact subset $K \subset \mathbb{R}^n$ and $\chi \in C_c^\infty(\mathbb{R}^n)$ there exists $C < \infty$ such that for all $s \in \mathbb{C}$ with* $\mathrm{Re}(s) \in [-1,0]$ *we have*

$$q_{K,\chi,p,r,r-2m}(\tilde{A}_s) \leq C(1+|s|)^2 \ .$$

Proof. We use (6) in order to reduce to the case $\mathrm{Re}(s) \in [-1,0]$. We have

$$\tilde{A}_{m,s} = q_m(s)^{-1}\Delta^m \hat{A}_s \ ,$$

where q_m is some polynomial of degree $2m-1$ which does not vanish if $\mathrm{Re}(s) \in [-1,0]$. Let $\chi' \in C_c^\infty$ be such that $\chi'\chi = \chi$. Since $\|\chi\Delta^m\|_{B(H^{p,r},H^{p,r-2m})} \le C$ we get for $f \in H_K^{p,r}$ with $\|f\|_{H^{p,r}=1}$

$$\begin{aligned} \|\chi\tilde{A}_{m,s}(f)\|_{H^{p,r-2m}} &= |q_m(s)^{-1}|\|\chi\Delta^m\hat{A}_s(f)\|_{H^{p,r-2m}} \\ &= |q_m(s)^{-1}|\|\chi\Delta^m\chi'\hat{A}_s(f)\|_{H^{p,r-2m}} \\ &\le |q_m(s)^{-1}|Cq_{K,\chi',p,r,r}(\hat{A}_s) \\ &\le C(1+|s|)^2 \end{aligned}$$

using Lemma 3.6 in the last step. □

We now prove the theorem. First we consider $s_0 \in \mathbb{C}\setminus\mathbb{N}_0$ such that $\mathrm{Re}(s_0) \ge 0$. Then there exists $m \in \mathbb{N}$ such that $\mathrm{Re}(s_0) \in [m-1,m]$. Let $K \subset \mathbb{R}^n$ be compact and $\chi \in C_c^\infty(\mathbb{R}^n)$. Let $f \in H_K^{p,r}$ be given. We must show that $\chi A_s f \in H^{p,r-2\mathrm{Re}(s_0)}$.

Assume first that $f \in C_c^\infty(\mathbb{R}^n)\cap H_K^{p,r}$. We consider the holomorphic family F_s with values in $S(\mathbb{R}^n)$ given by $F_s := (s+2)^{-3}f$ for $\mathrm{Re}(s) \in [-1,0]$. Then $\chi\tilde{A}_{m,s}F_s$ is a holomorphic family in $C_c^\infty(\mathbb{R}^n)$. Thus it is holomorphic with values in $H^{p,r-2m}$ and $\sup_{\mathrm{Re}(s)=[-1,0]}\|\chi\tilde{A}_{m,s}F_s\|_{H^{p,r-2m}} \le C\|f\|_{H^{p,r}}$ (see Lemma 3.7). Furthermore, we know that if $\mathrm{Re}(s)=0$, then $\chi\tilde{A}_{m-1,s}F_{s-1} \in H^{p,r-m+2}$ is uniformly bounded by $\|f\|_{H^{p,r}}$. Thus $\sup_{\mathrm{Re}(s)=-1}\|\chi\tilde{A}_{m,s}F_s\|_{H^{p,r-2m+2}} \le C\|f\|_{H^{p,r}}$. By complex interpolation we now conclude that

$$\begin{aligned} \chi\tilde{A}_{m,s_0-m}F_{s_0} &\in H^{p,r-2\mathrm{Re}(s_0)} \\ \|\chi\tilde{A}_{m,s_0-m}F_{s_0}\|_{H^{p,r-2\mathrm{Re}(s_0)}} &\le C\|f\|_{H^{p,r}} \end{aligned}$$

Note that any element of $f \in H_K^{p,r}$ can be approximated by smooth compactly supported functions. Since $\tilde{A}_{m,s_0-m}$ and F_{s_0} are multiples of A_{s_0} and f we conclude that

$$\chi A_{s_0} f \in H^{p,r-2\mathrm{Re}(s_0)} \ .$$

It remains to consider the integral points $m \in \mathbb{N}_0$. First of all note that $B_{m,-1}$ is a multiple of Δ^m. For $k \in \mathbb{N}_0$ and $f \in S'(\mathbb{R}^n)$ we write

$$B_{m,k}f = \frac{1}{2\pi\mathrm{i}}\int_{S(m,\epsilon)}\frac{1}{(s-m)^{k+1}}A_s f ds \ ,$$

where $S(m,\epsilon)$ is the circle with radius $\epsilon > 0$ centered at m, and the integral is taken in $S'(\mathbb{R}^n)$. If $f \in H^{p,r}$ and $\chi \in C_c^\infty(\mathbb{R}^n)$, then we have $\|\chi A_s f\|_{H^{p,r-2m-2\epsilon}} \le C$ uniformly on $S(m,\epsilon)$. For any $\phi \in C_c^\infty(\mathbb{R}^n)$ we can estimate

$$\begin{aligned}
|\langle \chi B_{m,k} f, \phi\rangle| &= |\frac{1}{2\pi \mathrm{i}} \int_{S(m,\epsilon)} \frac{1}{(s-m)^{k+1}} \langle \chi A_s f, \phi\rangle ds| \\
&\le \frac{1}{2\pi} \int_{S(m,\epsilon)} |\frac{1}{(s-m)^{k+1}} \langle \chi A_s f, \phi\rangle| ds \\
&\le \frac{1}{\epsilon^k} \sup_{s \in S(m,\epsilon)} \|\chi A_s f\|_{H^{p,r-2m-2\epsilon}} \|\phi\|_{H^{q,-r+2m+2\epsilon}} ds \\
&\le C \|\phi\|_{H^{q,-r+2m+2\epsilon}} \ .
\end{aligned}$$

We conclude that $\chi B_{m,k} f \in (H^{q,-r+2m+2\epsilon})^* = H^{p,r-2m-2\epsilon}$. Since $\epsilon > 0$ can be chosen arbitrary small we have $\chi B_{m,k} f \in H^{p,<(r-2m)}$ as asserted. □

We now apply this result to the spherical intertwining operator $\hat{J}^w_{1_\mu,\varphi}$. Let $\lambda \in \mathfrak{a}^*_\mathbb{C}$ be such that $\mathrm{Re}(\lambda) \ge 0$ and

$$\hat{J}^w_{1_\mu,\varphi} = \sum_l (\frac{\mu - \lambda}{\alpha})^l B_l$$

be the Laurent expansion of the meromorphic family $(\hat{J}^w_{1_\mu,\varphi})_\mu$ at λ. The coefficients have the following mapping property.

Corollary 3.8 *For all $p \in (1,\infty)$ and $r \in \mathbb{R}$*

$$B_l : H^{p,<r} \to H^{p,<r-2\mathrm{Re}(\lambda)} \ .$$

Proof. We fix a parabolic subgroup P, a maximal compact subgroup K, a Langlands decomposition $P = MAN$ with $M \subset K$, and a representative $w \in N_K(M)$ of the non-trivial element of the Weyl group $N_K(M)/M$. Let $G \setminus wP = \bar{N}MAN$, $g = \bar{n}(g)m(g)\alpha(g)\tilde{n}(g)$, be the Bruhat decomposition. Furthermore, we define $F : N \setminus \{1\} \to \bar{N} \setminus \{1\}$ such that $F(x)P = xwP$.

If $\mathrm{Re}(\lambda) < 0$ and $f \in C^\infty(\partial X, V(1_\lambda, \varphi))$, then by definition

$$\hat{J}^w_{1_\lambda,\varphi} f(g) = \int_{\bar{N}} f(gw\bar{x}) d\bar{x} \ .$$

We now assume that $\mathrm{supp}(f) \subset \bar{N}P$. We compute

$$(\hat{J}^w_{1_\lambda,\varphi} f)(1) = \int_{\bar{N}} f(w\bar{x}) d\bar{x}$$

$$
\begin{aligned}
&= \int_{\bar N} f(\bar x^w w) d\bar x \\
&= \int_N f(xw)dx \\
&= \int_{N\setminus\{1\}} \alpha(xw)^{\lambda-\rho} f(F(x))dx.
\end{aligned}
$$

We now identify the group N with its Lie algebra $\mathfrak{n}$ using the exponential map, and further we identify $\mathfrak{n}$ with $\mathbb{R}^{n-1}$ such that the action of M is orthogonal (this fixes the identification up to scale) and such that the measure dx identifies with the standard Lebesgue measure (this fixes the scale). The function $\alpha(xw)^\alpha$ is M-invariant and satisfies $\alpha(x^a w)^\alpha = a^{2\alpha}\alpha(xw)$, $a \in A$. Thus in the coordinates it is given by $\alpha(xw)^\alpha = c_1\|x\|^2$ for some constant $c_1 > 0$. We identify $\bar N$ with $\bar{\mathfrak{n}}$ using the exponential map, and furthermore, $\bar{\mathfrak{n}}$ with $\mathbb{R}^{n-1}$ using the isomorphism $\bar{\mathfrak{n}} \ni \bar X \mapsto X^w \in \mathfrak{n}$. The map $F : N\setminus\{0\} \to \bar N\setminus\{0\}$ is M-equivariant and satisfies $F(x^a) = F(x)^a$. Therefore, in our coordinates $F(x) = c_2 \frac{x}{\|x\|^2}$ for some $0 \neq c_2 \in \mathbb{R}$. So we can further compute

$$
\begin{aligned}
\int_{N\setminus\{1\}} \alpha(xw)^{\lambda-\rho} f(F(x))dx &= c_1^{2\lambda/\alpha-(n-1)} \int_{\mathbb{R}^{n-1}} \|x\|^{2\lambda/\alpha-(n-1)} f(c_2\frac{x}{\|x\|^2})dx \\
&= (c_1c_2)^{2\lambda/\alpha-(n-1)} \int_{\mathbb{R}^{n-1}} \|\bar y\|^{-2\lambda/\alpha-(n-1)} f(\bar y)d\bar y
\end{aligned}
$$

Let now $\bar z \in \bar N$. Then we define $f_{\bar z}(g) = f(\bar z g)$. Since $\hat J^w_{1_\lambda,\varphi}$ is G-equivariant, we have $(\hat J^w_{1_\lambda,\varphi} f)(\bar z) = (\hat J^w_{1_\lambda,\varphi} f_{\bar z})(1)$. In our coordinates we thus have

$$
\begin{aligned}
(\hat J^w_{1_\lambda,\varphi} f)(\bar z) &= (c_1c_2)^{2\lambda/\alpha-(n-1)} \int_{\mathbb{R}^{n-1}} \|\bar y\|^{-2\lambda/\alpha-(n-1)} f(\bar z + \bar y)d\bar y \\
&= (c_1c_2)^{2\lambda/\alpha-(n-1)} \int_{\mathbb{R}^{n-1}} \|\bar z - \bar y\|^{-2\lambda/\alpha-(n-1)} f(\bar y)d\bar y\ .
\end{aligned}
$$

Thus up to the harmless factor $(c_1c_2)^{2\lambda/\alpha-(n-1)}$ the intertwining operator coincides with A_s introduced at the beginning of the present subsection, where $s = \lambda/\alpha$. The assertion of the corollary now follows from Theorem 3.1 since we can cover ∂X by finitely many charts of the form $\bar N P$ (with varying P). $\square$

4 Proof of Theorem 1.15

4.1 Compatibility with twisting and embedding

We fix $\sigma \in \hat{M}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and a twist φ. If σ', μ, π and T is some twisting data, then we have the map

$$i_* : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^{-\infty}(\partial X, V(\sigma'_\mu, \varphi \otimes \pi)) \ .$$

Lemma 4.1 *Let $r \in \mathbb{R}$ and $p \in (1, \infty)$. If $\phi \in C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ satisfies $i_*\phi \in H^{p,r}(\partial X, V(\sigma'_\mu, \varphi \otimes \pi))$, then $\phi \in H^{p,r}(\partial X, V(\sigma_\lambda, \varphi))$.*

Proof. The map i_* is induced by an inclusion $V(\sigma_\lambda, \varphi) \hookrightarrow V(\sigma'_\mu, \varphi \otimes \pi)$ of vector bundles. Since we can find a complementary bundle of $V(\sigma_\lambda, \varphi)$ inside $V(\sigma'_\mu, \varphi \otimes \pi)$ there is also a projection from $V(\sigma'_\mu, \varphi \otimes \pi)$ to $V(\sigma_\lambda, \varphi)$. In local trivializations this projection is given by a matrix with smooth entries. Since smooth functions act continuously on the Sobolev space $H^{p,r}$ the projection provides a left-inverse of i_*. Now the lemma follows. $\square$

Now we discuss embedding. Let $G = G_m$, $\sigma_m \in \hat{M}_m$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. If $n > m$, σ_n and T is some embedding data, then we have the map

$$i_* : C^{-\infty}(\partial X^m, V^m((\sigma_m)_\lambda, \varphi)) \to C^{-\infty}(\partial X^n, V^n((\sigma_n)_{\lambda + \rho^m - \rho^n}, \varphi)) \ .$$

Lemma 4.2 *Let $r \in (-\infty, 0]$, $p \in (1, \infty)$, and q be given by $\frac{1}{p} + \frac{1}{q} = 1$. If $\phi \in C^{-\infty}(\partial X^m, V^m((\sigma_m)_\lambda, \varphi))$ satisfies $i_*\phi \in H^{p,<r-\frac{n-m}{q}}(\partial X^n, V^n((\sigma_n)_{\lambda+\rho^m-\rho^n}, \varphi))$, then $\phi \in H^{p,<r}(\partial X^m, V^m((\sigma_m)_\lambda, \varphi))$.*

Proof. The map T induces an inclusion of bundles

$$V^m((\sigma_m)_\lambda, \varphi) \hookrightarrow V^n((\sigma_n)_{\lambda+\rho^m-\rho^n}, \varphi)_{|\partial X^m} \ .$$

Let $E \to U$ be an extension of $V^m((\sigma_m)_\lambda, \varphi)$ to a tubular neighbourhood U of ∂X^m. It is now clear that $i_* = i_*^2 \circ i_*^1$ is a composition of the push-forward i_*^1 of distributions induced by the inclusion $\partial X^m \hookrightarrow \partial X^n$ of a regular submanifold, and the inclusion i_*^2 induced by the inclusion

of bundles. If $i_*\phi \in H^{p,<r-\frac{n-m}{q}}(\partial X^n, V^n((\sigma_n)_{\lambda+\rho^m-\rho^n},\varphi))$, then $i^1_*(\phi) \in H^{p,<r-\frac{n-m}{q}}(U,E)$ by Lemma 4.1. It remains to conclude that $\phi \in H^{p,<r}(\partial X^m, V^m((\sigma_m)_\lambda,\varphi))$.

Fix $\epsilon > 0$. We first assume that $n = m+1$. Let $l := [-r+\epsilon+\frac{1}{q}]$ be the integral part of $-r+\epsilon+\frac{1}{q}$. We define a restriction operator $R : C^\infty(U,E^+) \to \bigoplus_{i=0}^l C^\infty(\partial X^m, E^+_{|\partial X^m})$ by

$$R(f) := (f_{|\partial X^m}, (\partial_n f)_{|\partial X^m}, \dots, (\partial_n^l f)_{|\partial X^m}) \ ,$$

where ∂_n denotes the normal derivative, and $E^+ := E^* \otimes \Lambda$, Λ being the bundle of densities. We have an exact sequence

$$0 \to H_0^{q,-r+\epsilon+\frac{1}{q}}(U,E^+) \to H^{q,-r+\epsilon+\frac{1}{q}}(U,E^+) \xrightarrow{R} \bigoplus_{i=0}^{l} B^{-r+\epsilon-i}_{q,q}(\partial X^m, E^+_{|\partial X^m}) \to 0 \ ,$$

(see [15] for the definition of the Besov spaces and [15], 4.7.1, for the characterization of the range and the kernel of R). In particular, the space of smooth sections of E^+ vanishing to infinite order along ∂X^m is dense in $\ker R$.

Since $i^1_*(\phi) \in H^{p,r-\epsilon-\frac{1}{q}}(U,E)$ we know that $i^1_*\phi$ extends continuously to $H^{q,-r+\epsilon+\frac{1}{q}}(U,E^+)$. Furthermore, since $i^1_*\phi$ vanishes on the space of smooth sections of E^+ vanishing to infinite order along ∂X^m, we conclude that $i^1_*\phi$ vanishes on $H_0^{q,-r+\epsilon+\frac{1}{q}}(U,E^+)$ and thus factors over the quotient $\bigoplus_{i=0}^l B^{-r+\epsilon-i}_{q,q}(\partial X^m, E^+_{|\partial X^m})$. This means that ϕ extends continuously to $B^{-r+\epsilon}_{q,q}(\partial X^m, E^+_{|\partial X^m})$. We now use the continuous embedding ([15], 4.6.1)

$$H^{q,-r+2\epsilon} \hookrightarrow B^{-r+2\epsilon}_{q,\max(2,q)} \hookrightarrow B^{-r+\epsilon}_{q,q}$$

in order to conclude that ϕ restricts to a continuous functional on $H^{q,-r+2\epsilon}$. Since this holds for all $\epsilon > 0$ we have $\phi \in H^{p,<r}(\partial X^m, V^m((\sigma_m)_\lambda,\varphi))$. If $n-m > 1$, then we argue by induction. □

We now state the following stable version of our main theorem.

Theorem 4.3 *1. If $\phi \in \mathrm{Fam}^{st}_\Gamma(\Lambda_\Gamma,\sigma_\lambda,\varphi)$, then for p in a generic subset of $(1,\infty)$ there exist suitable twisting and embedding data such that $i_*\phi \in H^{p,<r_{p,\mu}(\Gamma)'}(\partial X', V(\sigma'_\mu,\varphi'))$, where we attach a "$'$" to objects which are associated to $\partial X'$ and φ'.*

2. If $\phi \in \mathrm{Cusp}_\Gamma(\sigma_\lambda,\varphi)$ and $\lambda \notin I_{\mathfrak{a}}$, then for all $p \in (1,\infty)$ there exist suitable twisting and embedding data such that $i_*\phi \in H^{p,<r^0_{p,\mu}(\Gamma)'}(\partial X', V(\sigma'_\mu,\varphi'))$.

The contents of this theorem is that given a stably deformable invariant distribution which is strongly supported on the limit set, or given a cusp form and $\lambda \notin I_{\mathfrak{a}}$, then after suitable embedding and twisting it belongs to the Sobolev space as asserted by Theorem 1.15 (for generic p).

Proposition 4.4 *Theorem 4.3 implies Theorem 1.15.*

Proof. Assume that i_* is induced by some embedding data. In this case $\varphi = \varphi'$, $\mu = \lambda + \rho - \rho'$, $\delta'_\Gamma = \delta_\Gamma + \rho - \rho'$. We observe that

$$\begin{aligned} r_{p,\lambda}(\Gamma)' &= r_{p,\mu}(\Gamma) - \frac{\dim(\partial X') - \dim(\partial X)}{q} \\ (\;\; r^0_{p,\lambda}(\Gamma)' &= r^0_{p,\mu}(\Gamma) - \frac{\dim(\partial X') - \dim(\partial X)}{q} \;\;) \,. \end{aligned}$$

If $r_{p,\lambda}(\Gamma) > 0$ ($r^0_{p,\lambda}(\Gamma) > 0$), then we employ the vanishing result [6] in order to conclude that $\phi = 0$. Else we combine the regularity of $i_*\phi$ asserted in Theorem 4.3 with Lemma 4.2 in order to see that ϕ has the regularity claimed in Theorem 1.15 provided p is generic or ϕ is a cusp form and $\lambda \notin I_{\mathfrak{a}}$.

Now assume that i_* is induced by some twisting data. In this case $\partial X = \partial X'$, $\rho = \rho'$, $\delta_\Gamma = \delta'_\Gamma$. Moreover, $\varphi' = \varphi \otimes \pi$. Let ν be the highest A-weight of π. Then $\mu - \nu = \lambda$ and $\delta_\varphi + \nu = \delta_{\varphi'}$. It is at this point where we use the assumption that π is irreducible. We observe that

$$\begin{aligned} r_{p,\lambda}(\Gamma)' &= r_{p,\mu}(\Gamma) \\ (\;\; r^0_{p,\lambda}(\Gamma)' &= r^0_{p,\mu}(\Gamma) \;\;) \,. \end{aligned}$$

Combining the regularity of $i_*\phi$ asserted in Theorem 4.3 with Lemma 4.1 we see that ϕ has the regularity claimed in Theorem 1.15 again provided that p is generic or ϕ is a cusp form and $\lambda \notin I_{\mathfrak{a}}$.

In the general case, where i_* is the composition of several embeddings and twistings, we argue by induction.

In order to drop the assumption that p is generic we argue as follows. Let $s < r_{p,\lambda}(\Gamma)$. Then we find $\tilde{p} > p$ such that Theorem 4.3 applies and $s < r_{\tilde{p},\lambda}(\Gamma)$. The argument above gives $\phi \in H^{\tilde{p},<r_{\tilde{p},\lambda}(\Gamma)}$ By the embedding theorem $H^{\tilde{p},<r_{\tilde{p},\lambda}(\Gamma)} \subset H^{p,s}$. Since $s < r_{p,\lambda}(\Gamma)$ was arbitrary,

we conclude that $\phi \in H^{p,<r_{p,\lambda}(\Gamma)}$. □

4.2 Sobolev regularity for cusps

In the present subsection we assume that $\Gamma = \Gamma_P$ for some (uniquely determined) Γ-cuspidal parabolic subgroup P. We assume that the cusp defined by Γ has smaller rank. We have a natural embedding

$$S_\Gamma(\sigma_\lambda, \varphi) \subset B_\Gamma(\sigma_\lambda, \varphi) \subset {}^\Gamma C^\infty(\Omega_\Gamma, V(\sigma_\lambda, \varphi)) \ .$$

Under certain conditions $f \in B_\Gamma(\sigma_\lambda, \varphi)$ ($f \in S_\Gamma(\sigma_\lambda, \varphi)$) defines a regular distribution $\hat{f} \in C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$. Our goal in the present subsection is to determine these conditions and to estimate the regularity of $\hat{f}$.

Let $N \subset P$ be the unipotent radical of P. Then we have the connected subgroup $N_\Gamma \subset N$ which is AP_Γ-invariant and such that $\Gamma\backslash N_\Gamma$ is compact. Here P acts on N by $(p, x) \mapsto p.x$ such that $pxwP = p.xwP$. Let $2\rho_\Gamma \in \mathfrak{a}^*$ denote the character of the action of A on $\Lambda^{max}\mathfrak{n}_\Gamma$, where $\mathfrak{n}_\Gamma$ denotes the Lie-algebra of N_Γ. This is consistent with the definition given earlier in the present paper. Furthermore, recall that $\rho^\Gamma := \rho - \rho_\Gamma$. Note that $\delta_\Gamma = -\rho^\Gamma$.

Let $S(N/N_\Gamma)$ be the unit sphere with respect to some choice of an euclidean M_Γ-invariant metric on the vector space N/N_Γ. Furthermore, we let U be the preimage of $S(N/N_\Gamma)^{A_+}$ under the projection $N \to N/N_\Gamma$. Then $Uw \subset \Omega_\Gamma$ is a Γ-invariant subset which projects onto a neighbourhood B^+_Γ of the cusp of B_Γ.

We fix a maximal compact subgroup K and a corresponding Langlands decomposition $P = MAN$. Let $w \in K$ be a representative of the non-trivial element of the Weyl group $N_K(M)/M$. Then we define the group $\bar{N} := N^w$ with Lie algebra $\bar{\mathfrak{n}}$. For $r \in \mathbb{N}_0$ let $(\bar{X}_j)_j$ be a base of $\mathcal{U}(\bar{\mathfrak{n}})^{\ge -r\alpha}$ (note that $X \in \bar{\mathfrak{n}}$ has degree $-\alpha$). If $\mathrm{supp}(f) \subset B^+_\Gamma$, then $\hat{f}$ is supported near ∞_P. It therefore makes sense to consider the norm

$$ {}_{\bar{N}}\|\hat{f}\|^p_{H^{p,r}} := \sum_j \int_{\bar{N}} |\hat{f}(\bar{x}\bar{X}_j)|^p d\bar{x} \ ,$$

where we have fixed a norm on V_φ.

Theorem 4.5 *Let $f \in B_\Gamma(\sigma_\lambda, \varphi)$ ($f \in S_\Gamma(\sigma_\lambda, \varphi)$) be supported in B^+_Γ.*

1. *If* $\rho^\Gamma > \delta_\varphi$ *(* *no condition* *) and* $\mathrm{Re}(\lambda) > -\rho^\Gamma + \delta_\varphi$*, then* f *determines a regular distribution* $\hat{f}$*.*

2. *If* $p \in (1,\infty)$ *and* $r, r^0 \in \mathbb{N}_0$ *satisfy*

$$r\alpha < \min\left(\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho+\rho^\Gamma}{p}\ ,\ -2\delta_\varphi - 2\rho_\Gamma + 2\frac{\rho}{p}\right)$$

$$\left(\quad r^0\alpha < \mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho+\rho^\Gamma}{p}\quad\right)\ ,$$

then we have $\hat{f} \in H^{p,r}(\partial X, V(\sigma_\lambda,\varphi))$ *(* $\hat{f} \in H^{p,r^0}(\partial X, V(\sigma_\lambda,\varphi))$ *).*

Proof. Note that $\hat{f}$ is smooth on Ω_Γ. In order to prove 1.) me must show that $\hat{f}$ is locally integrable near ∞_P. Since $\bar{N} \ni \bar{x} \mapsto \bar{x}P$ defines coordinates of ∂X near ∞_P it therefore suffices to show that

$$_{\bar{N}}\|\hat{f}\|_{H^{1,0}} := \int_{\bar{N}\setminus\{1\}} |f(\bar{x})| d\bar{x} < \infty\ . \tag{12}$$

Let $F : N\setminus\{1\} \to \bar{N}\setminus\{1\}$ be the diffeomorphism such that $F(x)m(xw)\alpha(xw)\tilde{n}(xw) = xw$ with $m(xw)\alpha(xw)\tilde{n}(xw) \in MAN$. We have

$$f(xw) = f(F(x)m(xw)\alpha(xw)\tilde{n}(xw)) = \sigma(m(xw))^{-1}\alpha(xw)^{\lambda-\rho} f(F(x))\ .$$

Thus $|f(F(x))| = |f(xw)|\alpha(xw)^{\rho-\mathrm{Re}(\lambda)}$. Furthermore $F^*d\bar{x} = |\det DF(x)|dx$. Note that $F(x^a) = F(x)^a$. From this we conclude that

$$|\det DF(x^a)| = |\det DF(x)| a^{-4\rho}\ . \tag{13}$$

Now (12) is equivalent to

$$\int_N |f(xw)|\alpha(xw)^{\rho-\mathrm{Re}(\lambda)}|\det DF(x)| dx < \infty\ .$$

Taking into account that $\mathrm{supp}(f) \in B_\Gamma^+$ we can write this integral in double polar coordinates as

$$\int_{A_+}\int_{S(N/N_\Gamma)}\int_A\int_{S(N_\Gamma)} |f(\eta^b\xi^a w)|\alpha(\eta^b\xi^a w)^{\rho-\mathrm{Re}(\lambda)}|\det DF(\eta^b\xi^a)| a^{2\rho^\Gamma} b^{2\rho_\Gamma} d\eta db d\xi da\ . \tag{14}$$

We choose some compact preimage $\hat{S}$ of $S(N/N_\Gamma)$ inside N.

Recall that φ extends to a semisimple representation of A. We can and will assume that V_φ decomposes into eigenspaces $V_\varphi(\beta)$, $\beta \in \mathfrak{a}_+^* \cup \{0\}$, w.r.t. the action of A so that $a \in A$ acts on $V_\varphi(\beta)$ by a^β, and such that $\max_{V_\varphi(\beta)\neq\{0\}} \beta = 2\delta_\varphi$. Let f_β be the component of f in

$V_\varphi(\beta)$. Furthermore, let $\varphi_{\beta_1,\beta_2}$ be the component of φ mapping $V_\varphi(\beta_2)$ to $V_\varphi(\beta_1)$. Note that $\varphi_{\beta_1,\beta_2}(p)=0$ if $\beta_2>\beta_1$ and $p\in AP_\Gamma$. Since f is Γ-invariant we have

$$f(\gamma xw)=\varphi(\gamma)f(xw)=\sum_{\beta_1\ge\beta_2}\varphi_{\beta_1,\beta_2}(\gamma)f_{\beta_2}(xw)\ .$$

We obtain the estimate

$$\sup_{\xi\in\hat{S},\eta\in S(N_\Gamma)}|f(\eta^b v^a w)|\le C\sum_{\beta_1\ge\beta_2}\sup_{v\in V}|f_{\beta_2}(\xi^a w)|\sup_{\eta\in S(N_\Gamma)}\|\varphi_{\beta_1,\beta_2}(\eta^b)\|\ ,$$

where $V\subset N\setminus N_\Gamma$ is a sufficiently large compact subset, and C is independent of a,b. Furthermore, using 2.5, $\alpha(x^a w)=a^2\alpha(xw)$, and (13), we get

$$\begin{aligned}\sup_{\eta\in S(N_\Gamma)}\|\varphi_{\beta_1,\beta_2}(\eta^b)\| &\le Cb^{\beta_1-\beta_2}\\ \sup_{v\in V}|f_{\beta_2}(v^a w)| &\le p^\Gamma_{V,1}(f)a^{2(\mathrm{Re}(\lambda)-\rho^\Gamma)+\beta_2} \qquad (15)\\ \sup_{\eta\in S(N_\Gamma),\xi\in\hat{S}}|\alpha(\eta^b\xi^a w)^{\rho-\mathrm{Re}(\lambda)}| &\le C\max(a,b)^{2(\rho-\mathrm{Re}(\lambda))}\\ \sup_{\eta\in S(N_\Gamma),\xi\in\hat{S}}|\det DF(\eta^b\xi^a)| &\le C\max(a,b)^{-4\rho}\ .\end{aligned}$$

Thus for $a\le b$ we have uniformly in (η,ξ)

$$\begin{aligned}&|f(\eta^b\xi^a w)|\alpha(\eta^b\xi^a w)^{\rho-\mathrm{Re}(\lambda)}|\det DF(\eta^b\xi^a)|a^{2\rho^\Gamma}b^{2\rho_\Gamma}\\ &\le\ C\sum_{\beta_1\ge\beta_2}b^{\beta_1-\beta_2}a^{2(\mathrm{Re}(\lambda)-\rho^\Gamma)+\beta_2}b^{2(\rho-\mathrm{Re}(\lambda))}b^{-4\rho}a^{2\rho^\Gamma}b^{2\rho_\Gamma}\ .\end{aligned}$$

Similarly, for $b\le a$ we have uniformly in (η,ξ)

$$\begin{aligned}&|f(\eta^b\xi^a w)|\alpha(\eta^b\xi^a w)^{\rho-\mathrm{Re}(\lambda)}|\det DF(\eta^b\xi^a)|a^{2\rho^\Gamma}b^{2\rho_\Gamma}\\ &\le\ C\sum_{\beta_1\ge\beta_2}b^{\beta_1-\beta_2}a^{2(\mathrm{Re}(\lambda)-\rho^\Gamma)+\beta_2}a^{2(\rho-\mathrm{Re}(\lambda))}a^{-4\rho}a^{2\rho^\Gamma}b^{2\rho_\Gamma}\end{aligned}$$

We now easily conclude that (14) converges if $\rho^\Gamma>\delta_\varphi$ and $\mathrm{Re}(\lambda)>-\rho^\Gamma+\delta_\varphi$.

If $f\in S_\Gamma(\sigma_\lambda,\varphi)$, then (15) can be improved to

$$\sup_{v\in V}|f_{\beta_2}(v^a w)|\le q^\Gamma_{W,1,k\alpha}(f)a^{2(\mathrm{Re}(\lambda)-\rho^\Gamma)+\beta_2-k\alpha}$$

for any $k\in\mathbb{N}_0$. We see that in this case (14) converges if $\mathrm{Re}(\lambda)>-\rho^\Gamma+\delta_\varphi$.

In order to prove assertions 2.) and 3.) of the theorem we extend the argument above. Let $\bar{X}\in\mathcal{U}(\mathfrak{n})$ be homogeneous of degree $-\nu\in\mathfrak{a}^*$, i.e. $\bar{X}^a=a^{-\nu}\bar{X}$ for $a\in A$. We then estimate

$$\int_{\bar{N}\setminus\{1\}}|f(\bar{x}\bar{X})|^p d\bar{x}\ . \qquad (16)$$

We again transform this integral into an integral over N. We define the function $X : N \setminus \{1\} \to \mathcal{U}(\mathfrak{n})$ so that the the equality $F(xX(x)) = F(x)\bar{X}$ of differential operators holds true. Let $X(x) = \sum_{\mu \le \nu} X_\mu(x)$ be its decomposition into homogeneous components. Because of

$$\begin{aligned} F(x^a X(x^a)) = F(x^a)\bar{X} &= F(x)^a \bar{X} = a^\nu (F(x)\bar{X})^a = a^\nu F(xX(x))^a \\ &= a^\nu F(x^a X(x)^a) = \sum_{\mu \le \nu} a^{\nu+\mu} F(x^a X_\mu(x)) \end{aligned}$$

we conclude that

$$X_\mu(x^a) = a^{\mu+\nu} X_\mu(x) \ .$$

Because of $\alpha(xw)^{\rho-\lambda}\sigma(m(xw))f(xw) = f(F(x))$ we must apply the Leibniz rule in order to express $f(F(x)\bar{X})$. We obtain a sum of terms of the form

$$h(x)\alpha(xY_1w)^{\rho-\lambda}\sigma(m(xY_2w))f(xY_3w) \ ,$$

where $Y_i \in \mathcal{U}(\mathfrak{n})$ is of degree d_i, $d_1 + d_2 + d_3 =: \mu \le \nu$, and $h(x^a) = a^{\mu+\nu}h(x)$ for $a \in A$. We must estimate

$$\int_N |h(x)\alpha(xY_1w)^{\rho-\lambda}\sigma(m(xY_2w))f(xY_3w)|^p |\det DF(x)| dx \ .$$

We again write this integral in the form

$$\begin{aligned} \int_{A_+} \int_{S(N/N_\Gamma)} \int_A \int_{S(N_\Gamma)} & |h(\eta^b\xi^a)\alpha(\eta^b\xi^a Y_1 w)^{\rho-\lambda}\sigma(m(\eta^b\xi^a Y_2 w))f(\eta^b\xi^a Y_3 w)|^p \\ & |\det DF(\eta^b\xi^a)| a^{2\rho^\Gamma} b^{2\rho_\Gamma} d\eta db d\xi da \ . \end{aligned} \tag{17}$$

Using the Γ-equivariance of f, $d_3 \le \nu \le k\alpha$ we can estimate

$$\begin{aligned} |f(\eta^b\xi^a Y_3 w)| &\le C \sum_{\beta_1 \ge \beta_2} \sup_{v \in V} |f_{\beta_2}(v^a Y_3 w)| \sup_{\eta \in S(N_\Gamma)} \|\varphi_{\beta_1,\beta_2}(\eta^b)\| \\ \sup_{v\in V} |f_{\beta_2}(v^a Y_3 w)| &\le p^\Gamma_{V,Y_3}(f) a^{2(\mathrm{Re}(\lambda)-\rho^\Gamma)+\beta_2-d_3} \\ |h(\eta^b\xi^a)| &\le C \max(a,b)^{\mu+\nu} \\ |\alpha(\eta^b\xi^a Y_1 w)^{\rho-\lambda}| &\le C \max(a,b)^{2(\rho-\mathrm{Re}(\lambda))-d_1} \\ |\sigma(m(\eta^b\xi^a Y_2 w))| &\le C \max(a,b)^{-d_2} \ . \end{aligned} \tag{18}$$

For $a \le b$ we have uniformly in (η, ξ)

$$\begin{aligned} & |h(\eta^b\xi^a)\alpha(\eta^b\xi^a Y_1 w)^{\rho-\lambda}\sigma(m(\eta^b\xi^a Y_2 w))f(\eta^b\xi^a Y_3 w)|^p |\det DF(\eta^b\xi^a)| a^{2\rho^\Gamma} b^{2\rho_\Gamma} \\ \le \ & C p^\Gamma_{V,Y_3}(f)^p \sum_{\beta_1 \ge \beta_2} b^{p(\mu+\nu)} b^{p[2(\rho-\mathrm{Re}(\lambda))-d_1]} b^{-pd_2} a^{p[2(\mathrm{Re}(\lambda)-\rho^\Gamma)+\beta_2-d_3]} b^{p(\beta_1-\beta_2)} b^{-4\rho} a^{2\rho^\Gamma} b^{2\rho_\Gamma} \ . \end{aligned}$$

Similarly for $b \le a$ we have uniformly in (η,ξ)

$$\begin{aligned} &|h(\eta^b\xi^a)\alpha(\eta^b\xi^aY_1w)^{\rho-\lambda}\sigma(m(\eta^b\xi^aY_2w))f(\eta^b\xi^aY_3w)|^p|\det DF(\eta^b\xi^a)|a^{2\rho^\Gamma}b^{2\rho_\Gamma} \\ \le\ & Cp^\Gamma_{V,Y_3}(f)^p\sum_{\beta_1\ge\beta_2} a^{p(\mu+\nu)}a^{p[2(\rho-\mathrm{Re}(\lambda))-d_1]}a^{-pd_2}a^{p[2(\mathrm{Re}(\lambda)-\rho^\Gamma)+\beta_2-d_3]}b^{p(\beta_1-\beta_2)}a^{-4\rho}a^{2\rho^\Gamma}b^{2\rho_\Gamma}\ . \end{aligned}$$

¿From these two estimates we conclude that (17) converges for $\nu < \min(\mathrm{Re}(\lambda)-\rho-\delta_\varphi+\frac{\rho+\rho^\Gamma}{p}, -2\delta_\varphi+2\rho^\Gamma-2\frac{\rho}{q})$. This implies assertion 2.) provided that $\hat f$ belongs to $H^{p,r}$.

If $f\in S_\Gamma(\sigma_\lambda,\varphi)$, then (18) can be improved to

$$\sup_{v\in V}|f_{\beta_2}(v^aY_3w)| \le q^\Gamma_{V,Y_3,k\alpha}(f)a^{2(\mathrm{Re}(\lambda)-\rho^\Gamma)+\beta_2-k\alpha}$$

for any $k\in\mathbb{N}_0$. We see that (17) now converges for $\nu<\mathrm{Re}(\lambda)-\rho-\delta_\varphi+\frac{\rho+\rho^\Gamma}{p}$. We thus obtain assertion 2.) in the case that f belongs to the Schwartz space again provided that $\hat f$ belongs to H^{p,r^0}.

In order to show that $\hat f\in H^{p,r}$ (resp. $\hat f\in H^{p,r^0}$) we must show that its distributional derivatives up to order r are indeed regular distributions. We proceed as follows. We construct a family of cut-off functions $\kappa_n\in C_c^\infty(\Omega_\Gamma)$, $n\in\mathbb{N}$, such that $\kappa_n\to 1$ locally uniformly on Ω_Γ as $n\to\infty$. We then have $\lim_{n\to\infty}\kappa_n\hat f=\hat f$ in the sense of distributions. For $\bar X\in\mathcal{U}(\bar{\mathfrak{n}})$ of degree $-\nu$, $\nu\le r\alpha$ we consider $(\kappa_n\hat f)(\bar x\bar X)=\kappa_n(x)\hat f(\bar x\bar X)+r_n(\bar x)$. Thus r_n subsumes all terms involving at least one derivative of κ_n. The estimate above already shows that $\kappa_n(x)\hat f(\bar x\bar X)$ converges in $H^{p,0}$ to $\widehat{f(.\bar X)}$. We will then show that r_n tends to zero in $H^{p,0}$. This proves that the distributional derivatives up to order r of $\hat f$ belong to $H^{p,0}$.

Let $\chi\in C^\infty(A)$ be a cut-off function such that $\chi(a)\equiv 1$ for $a\le 1$ and $\chi(a)\equiv 0$ for $a^\alpha\ge 2$. Furthermore, fix $1<a_0\in A$. Then we define $\tilde\kappa_n(x)\in C^\infty(N\setminus\{1\})$ by $\tilde\kappa_n(x):=\chi(\alpha(xw)a_0^{-n})$. Furthermore, we set $\kappa_n(xw):=\tilde\kappa_n(x)$. In order to estimate $\|r_n\|_{H^{p,r}}$ we proceed as above but we replace $f(xw)$ by $\tilde\kappa_n(x)f(xw)$. Then

$$(\kappa_nf)(\eta^b\xi^aY_3w)=\sum_k\tilde\kappa_n(\eta^b\xi^aZ'_k)f(\eta^b\xi^aZ_kw)$$

where $Z_k,Z'_k\in\mathcal{U}(\mathfrak{n})$, $\deg(Z_k)+\deg(Z'_k)=\deg(Y_3)$ are defined by the decomposition of the coproduct $\Delta(Y_3)=\sum_kZ_k\otimes Z'_k$. The terms which contribute to r_n have $\deg(Z'_k)>0$. In this case we have for $0<\epsilon\in\mathfrak{a}^*$

$$\begin{aligned} |\tilde\kappa_n(\eta^b\xi^aZ'_k)| &= \chi(\alpha(\eta^b\xi^aZ'_kw)a_0^{-n}) \\ &\le C\max(a,b)^{-\deg(Z'_k)+\epsilon}a_0^{-n\epsilon/2} \\ |f(\eta^b\xi^aZ_kw)| &\le C\sum_{\beta_1\ge\beta_2}a^{2(\mathrm{Re}(\lambda)-\rho^\Gamma)+\beta_2-\deg(Z_k)}b^{\beta_1-\beta_2}\ . \end{aligned} \tag{19}$$

In order to see (19) note that $\chi(\alpha(\eta^b\xi^a Z_k' w)a_0) \neq 0$ implies that $a_0\alpha(\eta^b\xi^a w) \in [1, a_1]$, where $a_1^\alpha = 2$. We can choose ϵ so small that $(\nu+\epsilon) < \min(\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho+\rho^\Gamma}{p}, -2\delta_\varphi + 2\rho^\Gamma - 2\frac{\rho}{q})$. Then we get a bound

$$\|r_n\|_{H^{p,r}} \le C a_0^{-n\epsilon/2} \ .$$

In the case that $f \in S_\Gamma(\sigma_\lambda, \varphi)$ the argument is similar. □

4.3 Sobolev regularity of invariant functions

In this subsection Γ is a general geometrically finite torsion-free subgroup of G. We assume that all cusps have smaller rank. Again, we have inclusions

$$S_\Gamma(\sigma_\lambda, \varphi) \subset B_\Gamma(\sigma_\lambda, \varphi) \subset {}^\Gamma C^\infty(\Omega_\Gamma, V(\sigma_\lambda, \varphi)) \ .$$

Under certain conditions $f \in B_\Gamma(\sigma_\lambda, \varphi)$ ($f \in S_\Gamma(\sigma_\lambda, \varphi)$) determines a regular distribution $\hat{f} \in C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$. Our goal ist to determine sufficient conditions for this to happen and to estimate the regularity of $\hat{f}$.

Let $\mathcal{P}$ denote the set of Γ-conjugacy classes of Γ-cuspidal parabolic subgroups. It parametrizes the set of cusps of Γ. For $P \in \mathcal{P}$ we define $\Gamma_P := \Gamma \cap P$. Then the results of Subsection 4.2 can be applied to Γ_P.

The main result of the present subsection is the following theorem.

Theorem 4.6 *Let $f \in B_\Gamma(\sigma_\lambda, \varphi)$ ($f \in S_\Gamma(\sigma_\lambda, \varphi)$).*

1. *If $\max_{[P]_\Gamma \in \mathcal{P}} \left(\delta_{\varphi_{|\Gamma_P}} - \rho^{\Gamma_P}\right) < 0$ (no condition) and $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$, then f determines a regular distribution $\hat{f}$.*

2. *If $p \in (1, \infty)$ and $r, r^0 \in \mathbb{N}_0$, $r \le r^0$,*

$$r\alpha < \min\left(\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho - \delta_\Gamma}{p}\ ,\ -2 \max_{[P]_\Gamma \in \mathcal{P}} \left[\delta_{\varphi_{|\Gamma_P}} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p}\right)$$

$$\left(\quad r^0\alpha < \mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho - \delta_\Gamma}{p} \quad\right),$$

then we have $\hat{f} \in H^{p,r}(\partial X, V(\sigma_\lambda, \varphi))$ ($\hat{f} \in H^{p,r^0}(\partial X, V(\sigma_\lambda, \varphi))$).

Proof. We first prove 1.). It suffices to show that

$$\int_{\Omega_\Gamma M} |f(k)| dk < \infty \ . \tag{20}$$

Let $Y_\Gamma := \Gamma \backslash X$ be the locally symmetric space of Γ and $\bar{Y}_\Gamma := \Gamma\backslash(X \cup \Omega_\Gamma)$ be its geodesic compactification. Let $P \in \mathcal{P}$ and $\bar{Y}_P$ be a representative of the end of $\bar{Y}_\Gamma$ corresponding to the cusp labeled by P such that we have an embedding $e_P : \bar{Y}_P \hookrightarrow \bar{Y}_{\Gamma_P}$. We choose a maximal compact subgroup K of G such that $wP \in \Omega_\Gamma$. Let $\mathcal{O} \in X$ denote the origin of X determined by this choice of K. We let $\Gamma^P \subset \Gamma$ be a set of representatives of $\Gamma_P\backslash\Gamma$, such that if $\gamma \in \Gamma^P$, then $\mathrm{dist}_X(\mathcal{O}, \gamma\mathcal{O}) \leq \mathrm{dist}(\mathcal{O}, \gamma'\mathcal{O})$ for all $\gamma' \in [\gamma]$ (comp. [4], Sec. 4.2). Let $B_P := B_\Gamma \cap \bar{Y}_P$. We first assume that $\mathrm{supp}(f) \subset B_P$. Since $\hat{f}$ is smooth in a neighbourhood of wP in order to prove (20) we must verify that

$$\int_{\bar{V}} |f(\bar{x})| d\bar{x} < \infty \ ,$$

where $\bar{V} \subset \bar{N}$ is a sufficiently large compact subset.

We consider the lift $(e_P^{-1})^* f =: f_P \in B_{\Gamma_P}(\sigma_\lambda, \varphi)$. Then we have

$$f = \sum_{\gamma \in \Gamma^P} \pi^{\sigma_\lambda, \varphi}(\gamma^{-1})(f_P)_{|\Omega_\Gamma}$$

as smooth functions on Ω_Γ, where the sum is locally finite. Since $\delta_{\varphi_{|\Gamma_P}} - \rho^{\Gamma_P} < 0$ (or f belongs to the Schwartz space) and $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi \geq \delta_{\varphi_{|\Gamma_P}} - \rho^{\Gamma_P}$ we already know by Theorem 4.5 that f_P determines a regular distribution $\widehat{(f_P)} \in L^1$. Let $G \setminus wP \ni g = \bar{n}(g)m(g)\alpha(g)\tilde{n}(g) \in \bar{N}MAN$ be the Bruhat decomposition of g. We now compute using $\frac{d\bar{n}(\gamma^{-1}\bar{x})}{d\bar{x}} = \alpha(\gamma^{-1}\bar{x})^{-2\rho}$ and $\alpha(\gamma\bar{n}(\gamma^{-1}\bar{x})) = \alpha(\gamma^{-1}\bar{x})^{-1}$

$$\begin{aligned}
\int_{\bar{V}} |f(\bar{x})| d\bar{x} &\leq \sum_{\gamma \in \Gamma^P} \int_{\bar{V}} |\varphi(\gamma)^{-1} f_P(\gamma\bar{x})| d\bar{x} \\
&= \sum_{\gamma \in \Gamma^P} \int_{\bar{V}} |\varphi(\gamma)^{-1} \alpha(\gamma\bar{x})^{\lambda-\rho} \sigma(m(\gamma\bar{x})^{-1}) f_P(\bar{n}(\gamma\bar{x}))| d\bar{x} \\
&= \sum_{\gamma \in \Gamma^P} \int_{\bar{V}} |\varphi(\gamma)^{-1} f_P(\bar{n}(\gamma\bar{x}))| \alpha(\gamma\bar{x})^{\mathrm{Re}(\lambda)-\rho} d\bar{x} \\
&= \sum_{\gamma \in \Gamma^P} \int_{\bar{n}(\gamma\bar{V})} |\varphi(\gamma)^{-1} f_P(\bar{x})| \alpha(\gamma\bar{n}(\gamma^{-1}\bar{x}))^{\mathrm{Re}(\lambda)-\rho} \frac{d\bar{n}(\gamma^{-1}\bar{x})}{d\bar{x}} d\bar{x} \\
&= \sum_{\gamma \in \Gamma^P} \int_{\bar{n}(\gamma\bar{V})} |\varphi(\gamma)^{-1} f_P(\bar{x})| \alpha(\gamma^{-1}\bar{x})^{-\mathrm{Re}(\lambda)-\rho} d\bar{x} \\
&\leq \sum_{\gamma \in \Gamma^P} \|\varphi(\gamma^{-1}\| \int_{\bar{N}} |f_P(\bar{x})| \alpha(\gamma^{-1}\bar{x})^{-\mathrm{Re}(\lambda)-\rho} d\bar{x} \ .
\end{aligned}$$

In order to proceed further we employ the following estimate. Let $g = k_g a_g h_g \in KA_+K$ denote the Cartan decomposition of g.

Lemma 4.7 *There is $c \in A_+$ such that $c^{-1}a_\gamma \le \alpha(\gamma^{-1}\bar{x}) \le c a_\gamma$ uniformly for all $\bar{x} \in \mathrm{supp}(f_P)$ and $\gamma \in \Gamma^P$.*

Assuming the lemma and using

$$\begin{aligned} \|\varphi(\gamma)^{-1}\| &\le C_\epsilon a_\gamma^{\delta_\varphi+\epsilon} \\ \int_{\bar{N}} |f_P(\bar{x})| d\bar{x} &< \infty \end{aligned}$$

for arbitrary small $\mathfrak{a}^* \ni \epsilon > 0$ we obtain

$$\int_{\bar{V}} |f(\bar{x})| d\bar{x} \le C \sum_{\gamma\in\Gamma^P} a_\gamma^{-\mathrm{Re}(\lambda)-\rho+\delta_\varphi+\epsilon} .$$

The sum on the right hand side is finite if $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi + \epsilon$.

This argument for all $P \in \mathcal{P}$ shows that $\hat{f}$ is a regular distribution if $\mathrm{supp}(f)$ is concentrated near the cusps of B_Γ.

Assume now that $f \in C_c^\infty(B_\Gamma, V_{B_\Gamma}(\sigma_\lambda, \varphi))$. We choose some cut-off function $\chi \in C_c^\infty(\Omega_\Gamma)$ such that $\sum_{\gamma\in\Gamma} \gamma^*\chi \equiv 1$ on $\mathrm{supp}(f)$. Then we can write

$$f = \sum_{\gamma\in\Gamma} \pi^{\sigma_\lambda,\varphi}(\gamma^{-1})(\chi f) .$$

We further choose any parabolic subgroup $P \subset G$ and a maximal compact subgroup K such that

$$wP \in \Omega_\Gamma \setminus \mathrm{supp}(f) . \tag{21}$$

This subgroup will be denoted by P_1 below. In order to prove that f defines a regular distribution $\hat{f}$ is suffices to show that

$$\int_{\bar{V}} |f(\bar{x})| d\bar{x} < \infty$$

for a sufficiently large compact subset $\bar{V} \subset \bar{N}$. We proceed in a similar manner as above. Define $\tilde{\chi}$ by $\tilde{\chi}(\bar{n}(g)) = \chi(g)$.

$$\int_{\bar{V}} |f(\bar{x})| d\bar{x} \le \sum_{\gamma\in\Gamma} \int_{\bar{V}} |\varphi(\gamma)^{-1}\chi(\gamma\bar{x}) f(\gamma\bar{x})| d\bar{x}$$

$$
\begin{aligned}
&= \sum_{\gamma\in\Gamma}\int_{\bar V} \tilde\chi(\bar n(\gamma\bar x))|\varphi(\gamma)^{-1}\alpha(\gamma\bar x)^{\lambda-\rho}\sigma(m(\gamma\bar x)^{-1})f(\bar n(\gamma\bar x))|d\bar x\\
&= \sum_{\gamma\in\Gamma}\int_{\bar V} \tilde\chi(\bar n(\gamma\bar x))|\varphi(\gamma)^{-1}f(\bar n(\gamma\bar x))|\alpha(\gamma\bar x)^{\mathrm{Re}(\lambda)-\rho}d\bar x\\
&= \sum_{\gamma\in\Gamma}\int_{\bar n(\gamma\bar V)} \tilde\chi(\bar x)|\varphi(\gamma)^{-1}f(\bar x)|\alpha(\gamma\bar n(\gamma^{-1}\bar x))^{\mathrm{Re}(\lambda)-\rho}\frac{d\bar n(\gamma^{-1}\bar x)}{d\bar x}d\bar x\\
&= \sum_{\gamma\in\Gamma}\int_{\bar n(\gamma\bar V)} \tilde\chi(\bar x)|\varphi(\gamma)^{-1}f(\bar x)|\alpha(\gamma^{-1}\bar x)^{-\mathrm{Re}(\lambda)-\rho}d\bar x\\
&\le \sum_{\gamma\in\Gamma}\|\varphi(\gamma^{-1}\|\int_{\bar N} \tilde\chi(\bar x)|f(\bar x)|\alpha(\gamma^{-1}\bar x)^{-\mathrm{Re}(\lambda)-\rho}d\bar x\ .
\end{aligned}
$$

We now use

Lemma 4.8 *There is $c\in A_+$ such that $c^{-1}a_\gamma\le\alpha(\gamma^{-1}\bar x)\le ca_\gamma$ uniformly for all $\bar x\in\mathrm{supp}(\chi f)$ and $\gamma\in\Gamma$.*

Again assuming the lemma and using

$$
\begin{aligned}
\|\varphi(\gamma)^{-1}\| &\le C_\epsilon a_\gamma^{\delta_\varphi+\epsilon}\\
\int_{\bar N}\tilde\chi(\bar x)|f(\bar x)|d\bar x &< \infty
\end{aligned}
$$

for arbitrary small $\mathfrak{a}^*\ni\epsilon>0$ we obtain

$$\int_{\bar V}|f(\bar x)|d\bar x\le C\sum_{\gamma\in\Gamma}a_\gamma^{-\mathrm{Re}(\lambda)-\rho+\delta_\varphi+\epsilon}\ .$$

As above the sum on the right hand side is finite if $\mathrm{Re}(\lambda)>\delta_\Gamma+\delta_\varphi+\epsilon$.

Any element $f\in B_\Gamma(\sigma_\lambda,\varphi)$ ($f\in S_\Gamma(\sigma_\lambda,\varphi)$) can be decomposed into a sum of functions which are either supported near the cusps of B_Γ or in the interior of B_Γ. Hence, we have shown the first part of the theorem asserting that $\hat f$ is a regular distribution.

We will now show $\hat f$ belongs to $H^{p,r}$ (H^{p,r^0}). So far we have shown that $\hat f\in H^{1,0}$. In order to obtain the stronger estimate we modify the proof above. We give the details in the case that $f\in\tilde B_\Gamma(\sigma_\lambda,\varphi)$.

Let $P\in\mathcal{P}$ and assume that $\mathrm{supp}(f)\subset B_P$. Let $\bar X\in\mathcal{U}(\bar{\mathfrak{n}})$ be homogeneous of degree $-\nu$ such that $\nu\le r\alpha$. We now must show that

$$\|f\|^p_{P,\bar V,\bar X,p}:=\int_{\bar V}|f(\bar x\bar X)|^pd\bar x<\infty$$

Let $\Delta : \mathcal{U}(\bar{\mathfrak{n}}) \to \mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathcal{U}(\bar{\mathfrak{n}})$ be the coproduct. We write $\Delta \otimes \mathrm{id}(\Delta(\bar{X})) = \sum_k A_k \otimes B_k \otimes C_k$. Then we compute

$$\begin{aligned}
&\int_{\bar{V}} |f(\bar{x}\bar{X})|^p d\bar{x} \\
&\leq \sum_{\gamma\in\Gamma^P} \int_{\bar{V}} |\varphi(\gamma)^{-1} f_P(\gamma\bar{x}\bar{X})|^p d\bar{x} \\
&= \sum_{\gamma\in\Gamma^P} \int_{\bar{V}} |\varphi(\gamma)^{-1} \sum_k \alpha(\gamma\bar{x}A_k)^{\lambda-\rho}\sigma(m(\gamma\bar{x}B_k)^{-1}) f_P(\bar{n}(\gamma\bar{x}C_k))|^p d\bar{x} \\
&\leq \sum_{\gamma\in\Gamma^P} \sum_k \int_{\bar{V}} \|\varphi(\gamma)^{-1}\|^p |f_P(\bar{n}(\gamma\bar{x}C_k))|^p \|\sigma(m(\gamma\bar{x}B_k)^{-1})\|^p |\alpha(\gamma\bar{x}A_k)^{\lambda-\rho}|^p d\bar{x} \\
&= \sum_{\gamma\in\Gamma^P} \sum_k \int_{\bar{n}(\gamma\bar{V})} \|\varphi(\gamma)^{-1}\|^p \|\sigma(m(\gamma\bar{n}(\gamma^{-1}\bar{x})B_k)^{-1})\|^p |\alpha(\gamma\bar{n}(\gamma^{-1}\bar{x})A_k)^{\lambda-\rho}|^p \\
&\qquad\qquad |f_P(\bar{n}(\gamma\bar{n}(\gamma^{-1}\bar{x})C_k))|^p \alpha(\gamma^{-1}\bar{x})^{-2\rho} d\bar{x}\ .
\end{aligned}$$

We compute

$$\begin{aligned}
m(\gamma\bar{n}(\gamma^{-1}\bar{x})B_k) &= m(\gamma\gamma^{-1}\bar{x}\tilde{n}(\gamma^{-1}\bar{x})^{-1}\alpha(\gamma^{-1}\bar{x})^{-1}m(\gamma^{-1}\bar{x})^{-1}B_k) \\
&= m(\gamma^{-1}\bar{x})^{-1} m(B_k^{[\tilde{n}(\gamma^{-1}\bar{x})^{m(\gamma^{-1}\bar{x})}]^{-1}\alpha(\gamma^{-1}\bar{x})^{-1}}) \\
&= \alpha(\gamma^{-1}\bar{x})^{-\deg(B_k)} m(\gamma^{-1}\bar{x})^{-1} m(B_k^{[\tilde{n}(\gamma^{-1}\bar{x})^{m(\gamma^{-1}\bar{x})}]^{-1}}) \\
\alpha(\gamma\bar{n}(\gamma^{-1}\bar{x})A_k) &= \alpha(\gamma\gamma^{-1}\bar{x}\tilde{n}(\gamma^{-1}\bar{x})^{-1}\alpha(\gamma^{-1}\bar{x})^{-1}m(\gamma^{-1}\bar{x})^{-1}A_k) \\
&= \alpha(\gamma^{-1}\bar{x})^{-\deg(A_k)} \alpha(A_k^{[\tilde{n}(\gamma^{-1}\bar{x})^{m(\gamma^{-1}\bar{x})}]^{-1}}) \\
\bar{n}(\gamma\bar{n}(\gamma^{-1}\bar{x})C_k) &= \bar{n}(\gamma\gamma^{-1}\bar{x}\tilde{n}(\gamma^{-1}\bar{x})^{-1}\alpha(\gamma^{-1}\bar{x})^{-1}m(\gamma^{-1}\bar{x})^{-1}C_k) \\
&= \alpha(\gamma^{-1}\bar{x})^{-\deg(C_k)} \bar{x}\bar{n}(C_k^{[m(\gamma^{-1}\bar{x})\tilde{n}(\gamma^{-1}\bar{x})]^{-1}})\ .
\end{aligned}$$

We now employ the following lemma.

Lemma 4.9 *There exists a compact subset $U \subset N$ such that $\tilde{n}(\gamma^{-1}\bar{x}) \in U$ for all $\bar{x} \in \mathrm{supp}(f_P)$ and $\gamma \in \Gamma^P$.*

Assuming the lemma for a moment we obtain uniformly for $\bar{x} \in \mathrm{supp}(f_P)$ and $\gamma \in \Gamma^P$

$$\begin{aligned}
\|\sigma(m(\gamma\bar{n}(\gamma^{-1}\bar{x})B_k)^{-1})\| &\leq C\alpha(\gamma^{-1}\bar{x})^{-\deg(B_k)} \\
|\alpha(\gamma\bar{n}(\gamma^{-1}\bar{x})A_k)^{\lambda-\rho}| &\leq C\alpha(\gamma^{-1}\bar{x})^{-\deg(A_k)}\alpha(\gamma\bar{x})^{-\mathrm{Re}(\lambda)+\rho}\ .
\end{aligned}$$

Furthermore, using Theorem 4.5 we have uniformly for $\gamma \in \Gamma^P$

$$\int_{\bar{N}} |f_P(\bar{n}(\gamma\bar{n}(\gamma^{-1}\bar{x})C_k))|^p d\bar{x}$$

$$\begin{aligned}
&\leq C \sup_{\bar{x}\in\mathrm{supp}(f_P)} \alpha(\gamma^{-1}\bar{x})^{-p\deg(C_k)}{}_{\bar{N}}\|\widehat{f_P}\|^p_{H^{p,\frac{\deg(C_k)}{\alpha}}}\\
&\leq C' \sup_{\bar{x}\in\mathrm{supp}(f_P)} \alpha(\gamma^{-1}\bar{x})^{-p\deg(C_k)}\ .
\end{aligned}$$

Using in addition Lemma 4.7 we find

$$\|f\|_{P,\bar{V},X,p} \leq C_\epsilon \left(\sum_{\gamma\in\Gamma^P}\sum_k a_\gamma^{p(-\deg(C_k)-\deg(B_k)-\deg(A_k)-\mathrm{Re}(\lambda)+\rho+\delta_\varphi+\epsilon)-2\rho}\right)^{\frac{1}{p}}$$

for arbitrary small $\epsilon > 0$. Since $-\deg(C_k)-\deg(B_k)-\deg(A_k) = -\mu \leq r\alpha$ the sum over Γ^P on the right hand side is finite if

$$r\alpha < \mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho-\delta_\Gamma}{p}\ .$$

We conclude that $\hat{f} \in H^{p,r}$ if $f \in \tilde{B}_\Gamma(\sigma_\lambda,\varphi)$ is supported near the cusps.

In the case that f belongs to the Schwartz space the obvious modification of the argument above yields $\hat{f} \in H^{p,r^0}$.

Assume now that $f \in C_c^\infty(B_\Gamma, V(\sigma_\lambda,\varphi))$. We choose the parabolic subgroup $P \subset G$ and the maximal compact subgroup K such that (21) holds. We must show that

$$\|\hat{f}\|^p_{P,V,\bar{X},p} := \int_{\bar{V}} |f(\bar{x}\bar{X})|^p d\bar{x} < \infty\ .$$

We compute

$$\begin{aligned}
&\int_{\bar{V}} |f(\bar{x}\bar{X})|^p d\bar{x}\\
&\leq \sum_{\gamma\in\Gamma}\int_{\bar{V}} |\varphi(\gamma)^{-1}(\chi f)(\gamma\bar{x}\bar{X})|^p d\bar{x}\\
&= \sum_{\gamma\in\Gamma}\int_{\bar{V}} |\sum_k \varphi(\gamma)^{-1}\alpha(\gamma\bar{x}A_k)^{\lambda-\rho}\sigma(m(\gamma\bar{x}B_k)^{-1})(\tilde{\chi}f)(\bar{n}(\gamma\bar{x}C_k))|^p d\bar{x}\\
&\leq \sum_{\gamma\in\Gamma}\sum_k\int_{\bar{V}} \|\varphi(\gamma)^{-1}\|^p|(\tilde{\chi}f)(\bar{n}(\gamma\bar{x}C_k))|^p\|\sigma(m(\gamma\bar{x}B_k)^{-1})\|^p|\alpha(\gamma\bar{x}A_k)^{\lambda-\rho}|^p d\bar{x}\\
&= \sum_{\gamma\in\Gamma}\sum_k\int_{\bar{n}(\gamma\bar{V})} \|\varphi(\gamma)^{-1}\|^p\|\sigma(m(\gamma\bar{n}(\gamma^{-1}\bar{x})B_k)^{-1})\|^p|\alpha(\gamma\bar{n}(\gamma^{-1}\bar{x})A_k)^{\lambda-\rho}|^p\\
&\qquad |(\tilde{\chi}f)(\bar{n}(\gamma\bar{n}(\gamma^{-1}\bar{x})C_k))|^p\alpha(\gamma^{-1}\bar{x})^{-2\rho}d\bar{x}\ .
\end{aligned}$$

Lemma 4.10 *There exists a compact subset $U \subset N$ such that $\tilde{n}(\gamma^{-1}\bar{x}) \in U$ for all $\bar{x} \in \mathrm{supp}(\chi f)$ and $\gamma \in \Gamma$.*

Using this lemma we conclude that for arbitrary small $\epsilon > 0$

$$\|f\|^p_{P,\bar{V},\bar{X},p} \le C_\epsilon \sum_{\gamma\in\Gamma}\sum_k a_\gamma^{p(-\deg(C_k)-\deg(B_k)-\deg(A_k)-\mathrm{Re}(\lambda)+\rho+\delta_\varphi+\epsilon)-2\rho}$$

for arbitrary small $\epsilon > 0$. Note that $\deg(C_k)+\deg(B_k)+\deg(A_k) = \deg(X)$. We conclude that $\hat{f} \in H^{p,r^0}$.

Again, since any element $f \in B_\Gamma(\sigma_\lambda, \varphi)$ ($f \in S_\Gamma(\sigma_\lambda, \varphi)$) can be decomposed into a sum of functions which are either supported near the cusps of B_Γ or in the interior of B_Γ we have shown that $\hat{f} \in H^{p,r}$ ($\hat{f} \in H^{p,r^0}$), if $r \in \mathbb{N}_0$ ($r^0 \in \mathbb{N}_0$) satisfy the assumptions of the theorem.

Proof. [Lemma 4.7] For $g \in G$ let $g = \kappa(g)a(g)n(g) \in KAN$ be the Iwasawa decomposition. Furthermore, if $g \in G \setminus wP$, then we have the Bruhat decomposition $g = \bar{n}(g)m(g)\alpha(g)\tilde{n}(g) \in \bar{N}MAN$. In particular,

$$\begin{aligned}
\kappa(g)a(g)n(g) &= \bar{n}(g)m(g)\alpha(g)\tilde{n}(g) \\
&= \kappa(\bar{n}(g))a(\bar{n}(g))n(\bar{n}(g))m(g)\alpha(g)\tilde{n}(g) \\
&= \kappa(\bar{n}(g))m(g)a(\bar{n}(g))\alpha(g)n(\bar{n}(g))^{\alpha(g)^{-1}m(g)^{-1}}\tilde{n}(g) \ ,
\end{aligned}$$

so that $a(g) = a(\bar{n}(g))\alpha(g)$. Since $a(\bar{n}) \ge 1$ we conclude that $\alpha(g) \le a(g)$.

Note that there is at most one $\gamma_0 \in \Gamma^P$ such that $wP \in \gamma^{-1}\mathrm{supp}(f_P)$. If this element exists, then we define $\Gamma^P_0 := \Gamma^P \setminus \{\gamma_0\}$. If not, then we set $\Gamma^P_0 := \Gamma^P$.

There is a compact subset $V \subset \bar{N}$ such that $\bar{n}(\gamma^{-1}\bar{x}) \in V$ for all $\bar{x} \in \mathrm{supp}(f_P)$ and $\gamma \in \Gamma^P_0$. In particular, we have $a(\bar{n}(\gamma^{-1}\bar{x})) \le c_1$ for some $c_1 \in A$. We conclude that

$$c_1^{-1}a(\gamma^{-1}\bar{x}) \le \alpha(\bar{n}(\gamma^{-1}\bar{x})) \le a(\gamma^{-1}\bar{x}) \ .$$

Furthermore note that $a(\gamma^{-1}\bar{x}) = a(\gamma^{-1}\kappa(\bar{x}))a(\bar{x})$, and $1 \le a(\bar{x}) \le c_2$ for all $\bar{x} \in \mathrm{supp}(f_P)$. Thus

$$c_1^{-1}a(\gamma^{-1}\kappa(\bar{x})) \le \alpha(\bar{n}(\gamma^{-1}\bar{x})) \le c_2 a(\gamma^{-1}\kappa(\bar{x})) \ . \tag{22}$$

We now want to apply Lemma [5], Lemma 2.3 asserting that there is $c_3 \in A_+$ such that for all $\gamma \in \Gamma^P$ and $\bar{x} \in \mathrm{supp}(f_P)$

$$c_3^{-1}a_\gamma \le a(\gamma^{-1}\kappa(\bar{x})) \le a_\gamma \ . \tag{23}$$

In order to verify the assumption of this Lemma we must show that $WA_+K\cap\Gamma^P$ is finite, where W is some neighbourhood of $\kappa(\mathrm{supp}(f_P))M$. Note that $\mathrm{supp} f_P \subset \Gamma_P e_P(B_P)$. We choose W such that $\kappa(\mathrm{supp}(f_P))M \subset W \subset \kappa(\Gamma_P e_P(B_P))M$.

We argue by contradiction. So assume that $WA_+K\cap\Gamma^P$ is infinite. Then we can find a sequence $\gamma_i \in WA_+K\cap\Gamma^P$ such that $\gamma_i \to \Lambda_{\Gamma^P}$, where Λ_{Γ^P} denotes the set of accumulation points in ∂X of $\Gamma^P\mathcal{O}$. Let $D(\mathcal{O},\Gamma_P)$ be the Dirichlet domain of Γ_P with respect to the origin. By construction of Γ^P we have $\Lambda_{\Gamma^P} \subset \overline{D(\mathcal{O},\Gamma_P)} \setminus \Gamma_P e_P(\bar{Y}_P)$. Since $W \subset \kappa(\Gamma_P e_P(B_P))M$ we have $\gamma_i \notin WA_+K$ for $i >> 0$. This is the contradiction.

Combining (22) and (23) we obtain the assertion of Lemma 4.7. □

Proof. [Lemma 4.8] Note that there is at most one $\gamma_0 \in \Gamma$ such that $wP \in \gamma^{-1}\mathrm{supp}(\chi f)$. If this element exists then we define $\Gamma_0 := \Gamma \setminus \{\gamma_0\}$. If not, then we set $\Gamma_0 := \Gamma$.

There is a compact subset $V \subset \bar{N}$ such that $\bar{n}(\gamma^{-1}\bar{x}) \in V$ for all $\bar{x} \in \mathrm{supp}(\chi f)$ and $\gamma \in \Gamma_0$. We argue as in the proof of Lemma 4.7 that there is $c_1 \in A_+$ such that

$$c_1^{-1} a(\gamma^{-1}\kappa(\bar{x})) \le \alpha(\bar{n}(\gamma^{-1}\bar{x})) \le c_1 a(\gamma^{-1}\kappa(\bar{x})) \ . \tag{24}$$

We again want to apply Lemma [5], Lemma 2.3, which gives $c_2 \in A_+$ such that for all $\gamma \in \Gamma$ and $\bar{x} \in \mathrm{supp}(\chi f)$

$$c_2^{-1} a_\gamma \le a(\gamma^{-1}\kappa(\bar{x})) \le a_\gamma \ . \tag{25}$$

In order to verify the assumption of this Lemma we must show that $WA_+K\cap\Gamma$ is finite, where W is some neighbourhood of $\kappa(\mathrm{supp}(\chi f))M$. Note that $\mathrm{supp}(\chi f) \subset \Omega_\Gamma$. We choose a compact W such that $\kappa(\mathrm{supp}(\chi f))M \subset W \subset \kappa(\Omega_\Gamma)M$.

We again argue by contradiction. So assume that $WA_+K\cap\Gamma$ is infinite. Then we can find a sequence $\gamma_i \in WA_+K\cap\Gamma$ such that $\gamma_i \to \Lambda_\Gamma$. Since $W \subset \kappa(\Omega_\Gamma)M$ we have $\gamma_i \notin WA_+K$ for $i >> 0$. This is the contradiction.

Combining (24) and (25) we obtain the assertion of Lemma 4.8. □

Proof. [Lemma 4.9] We have

$$\begin{aligned}(\gamma^{-1}\bar{x})^{-1}wP &= \tilde{n}(\gamma^{-1}\bar{x})^{-1}\alpha(\gamma^{-1}\bar{x})^{-1}m(\gamma^{-1}\bar{x})^{-1}\bar{n}(\gamma^{-1}\bar{x})^{-1}wP \\ &= \tilde{n}(\gamma^{-1}\bar{x})^{-1}wP\end{aligned}$$

Assume that there are sequences $\bar{x}_i \in \operatorname{supp}(f_P)$ and $\gamma_i \in \Gamma^P$ such that $\tilde{n}(\gamma_i^{-1}\bar{x}_i) \to \infty$. Then $\lim_{i\to\infty} \bar{x}_i^{-1}\gamma_i wP = \infty_P$. Taking a subsequence we can assume that $\bar{x}_i$ converges to $\bar{x}_0 \in \operatorname{supp}(f_P)$. Then $\lim_{i\to\infty}\gamma_i wP = \bar{x}_0\infty_P \in \operatorname{supp}(f_P)$. This is impossible since $\Lambda_{\Gamma^P}\cap\operatorname{supp}(f_P) = \emptyset$. □

Proof. [Lemma 4.10] The proof is similar to that of Lemma 4.9. Assume that there are sequences $\bar{x}_i \in \operatorname{supp}(\chi f)$ and $\gamma_i \in \Gamma$ such that $\tilde{n}(\gamma_i^{-1}\bar{x}_i) \to \infty$. Then $\lim_{i\to\infty} \bar{x}_i^{-1}\gamma_i wP = \infty_P$. Taking a subsequence we can assume that $\bar{x}_i$ converges to $\bar{x}_0 \in \operatorname{supp}(\chi f)$. Then $\lim_{i\to\infty}\gamma_i wP = \bar{x}_0\infty_P \in \operatorname{supp}(f_P)$. This is impossible since $\Lambda_\Gamma \cap \operatorname{supp}(\chi f) = \emptyset$. □

We now use the freedom to choose the exponent p in the assertion of Theorem 4.6 in order to extend the result to fractional Sobolev order.

Corollary 4.11 *1. Let $f \in B_\Gamma(\sigma_\lambda,\varphi)$ and assume that $\max_{[P]_\Gamma\in\mathcal{P}}\left(\delta_{\varphi|\Gamma_P} - \rho^{\Gamma_P}\right) < 0$ and* $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$*. In addition we assume that for $p \in (1,\infty)$ we have that*

$$\epsilon := \left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho-\delta_\Gamma}{p}\right] - \left[-2\max_{[P]_\Gamma\in\mathcal{P}}\left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p}\right]$$

satisfies $|\frac{\epsilon}{\alpha}| > \frac{\rho+\delta_\Gamma}{\rho-\delta_\Gamma}$, $\frac{\rho-\delta^\Gamma}{p} > \alpha$, $\frac{\rho-\delta^\Gamma}{q} > \alpha$, and there is an integer $k\in\mathbb{N}$ such that

$$0 < \left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho-\delta_\Gamma}{p}\right] < k\alpha \le \left[-2\max_{[P]_\Gamma\in\mathcal{P}}\left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p}\right] \quad \text{if } \epsilon < 0 \quad (26)$$

$$0 < \left[-2\max_{[P]_\Gamma\in\mathcal{P}}\left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p}\right] < k\alpha \le \left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho-\delta_\Gamma}{p}\right] \quad \text{if } \epsilon > 0 \ .(27)$$

Then $\hat{f} \in H^{p,r}$ provided $r\in\mathbb{R}$ satisfies

$$r\alpha < \min\left(\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho-\delta_\Gamma}{p}\ ,\ -2\max_{[P]_\Gamma\in\mathcal{P}}\left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p}\right)\ .$$

2. Let $f \in S_\Gamma(\sigma_\lambda,\varphi)$ and assume that $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$*. In addition we assume that for $p \in (1,\infty)$*

we have $\frac{\rho-\delta_\Gamma}{p} > \alpha$, $\frac{\rho-\delta^\Gamma}{q} > \alpha$, *and* $\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho-\delta_\Gamma}{p} > 0$. *Then* $\hat{f} \in H^{p,r^0}$ *provided* $r^0 \in \mathbb{R}$ *satisfies*

$$r^0\alpha < \mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho - \delta_\Gamma}{p} .$$

Proof. We begin with the first assertion in case $\epsilon > 0$. Let $k \in \mathbb{N}$ be the minimal number such that (27) holds true. For sufficiently small $\delta > 0$ we define $1 < p_0 < p < p_1 < \infty$ such that

$$\begin{aligned} (k-1)\alpha + \delta &= \left[-2 \max_{[P]_\Gamma \in \mathcal{P}} \left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p_1}\right] \\ k\alpha + \delta &= \left[-2 \max_{[P]_\Gamma \in \mathcal{P}} \left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p_0}\right] . \end{aligned}$$

If δ is sufficiently small, then we have

$$\begin{aligned} (k-1)\alpha &< \min\left[\left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho-\delta_\Gamma}{p_1}\right], \left[-2 \max_{[P]_\Gamma \in \mathcal{P}} \left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p_1}\right]\right] \\ k\alpha &< \min\left[\left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho-\delta_\Gamma}{p_0}\right], \left[-2 \max_{[P]_\Gamma \in \mathcal{P}} \left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p_0}\right]\right] . \end{aligned}$$

We now apply Theorem 4.6 in order to conclude that $\hat{f} \in H^{p_0,k} \cap H^{p_1,k-1}$. We define $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1}{p_1} + \theta(\frac{1}{p_0} - \frac{1}{p_1})$. Then interpolation gives $\hat{f} \in H^{p,s}$ for $s := k - 1 + \theta$. Note that s depends on δ. If we choose $\delta > 0$ sufficiently small, then $s > r$, and the assertion of the corollary follows.

Let now $\epsilon < 0$. Let $k \in \mathbb{N}$ be the minimal number such that (26) holds true. For sufficiently small $\delta > 0$ we define $1 < p_0 < p < p_1 < \infty$ such that

$$\begin{aligned} (k-1)\alpha + \delta &= \left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho - \delta_\Gamma}{p_1}\right] \\ k\alpha + \delta &= \left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho - \delta_\Gamma}{p_0}\right] . \end{aligned}$$

If δ is sufficiently small, then we have

$$\begin{aligned} (k-1)\alpha &< \min\left[\left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho-\delta_\Gamma}{p_1}\right], \left[-2 \max_{[P]_\Gamma \in \mathcal{P}} \left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p_1}\right]\right] \\ k\alpha &< \min\left[\left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho-\delta_\Gamma}{p_0}\right], \left[-2 \max_{[P]_\Gamma \in \mathcal{P}} \left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p_0}\right]\right] . \end{aligned}$$

We now apply Theorem 4.6 in order to conclude that $\hat{f} \in H^{p_0,k} \cap H^{p_1,k-1}$. We define $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1}{p_1} + \theta(\frac{1}{p_0} - \frac{1}{p_1})$. Then interpolation gives $\hat{f} \in H^{p,s}$ for $s := k - 1 + \theta$. If we choose $\delta > 0$ sufficiently small, then $s > r$, and the assertion of the corollary follows again.

Now we consider the case that $f \in S_\Gamma(\sigma_\lambda, \varphi)$. For sufficiently small $\delta > 0$ we define $1 < p_0 < p < p_1 < \infty$ such that

$$\begin{aligned}(k-1)\alpha + \delta &= \left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho - \delta_\Gamma}{p_1}\right] \\ k\alpha + \delta &= \left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho - \delta_\Gamma}{p_0}\right] ,\end{aligned}$$

where $k \in \mathbb{N}$ is uniquely determined. We now apply Theorem 4.6 in order to conclude that $\hat{f} \in H^{p_0,k} \cap H^{p_1,k-1}$. We define $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1}{p_1} + \theta(\frac{1}{p_0} - \frac{1}{p_1})$. Then interpolation gives $\hat{f} \in H^{p,s}$ for $s := k - 1 + \theta$. If we choose $\delta > 0$ sufficiently small, tean $s > r$, and the assertion of the corollary follows in this case, too. □

We now deduce the following consequence of Corollary 4.11.

Corollary 4.12 *Let $p \in (1,\infty)$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ satisfy the assumption of Corollary 4.11, 1. Furthermore, let $(f_\mu)_\mu$, $f_\mu \in B_\Gamma(\sigma_\mu, \varphi)$, be a germ at λ of a meromorphic family and $\hat{f}_\mu = \sum_l (\mu - \lambda)^l \hat{f}_l$ be the Laurent expansion at λ of $(\hat{f}_\mu)_\mu$. Then $\hat{f}_l \in H^{p,<r}$, where*

$$r\alpha = \min\left(\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho - \delta_\Gamma}{p} , -2 \max_{[P]_\Gamma \in \mathcal{P}} \left[\delta_{\varphi|\Gamma_P} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p}\right) .$$

Proof. Choose $\epsilon > 0$ sufficiently small such that $F_\mu := (\mu - \lambda)^{-\mathrm{ord}_{\mu=\lambda} f_\mu} f_\mu$ is defined and holomorphic on $\{|\mu - \lambda| \le \epsilon\}$, and $r - 2\epsilon > 0$. The proofs of Theorem 4.6 and Corollary 4.11 show that $\|F_\mu\|_{H^{p,r-2\epsilon}}$ is locally bounded. Since $\hat{F}_\mu = ext^\Gamma F_\mu$ is a germ of a holomorphic family of distributions we conclude that F_μ is weakly continuous in $H^{p,r-2\epsilon}$. Since the embedding $H^{p,r-2\epsilon} \hookrightarrow H^{p,r-3\epsilon}$ is compact the family F_μ is holomorphic with values in $H^{p,r-3\epsilon}$. The coefficients $\hat{f}_l$ can be expressed as

$$\hat{f}_l = \frac{1}{2\pi \mathrm{i}} \int_{|\mu-\lambda|=\epsilon} (\mu - \lambda)^{\mathrm{ord}_{\mu=\lambda} f_\mu - l - 1} F_\mu ,$$

and therefore $\hat{f}_l \in H^{p,r-3\epsilon}$. Since $\epsilon > 0$ can be chosen arbitrarily small the assertion follows. □

4.4 Proof of Theorem 4.3

We begin with the first assertion of the theorem. For a generic set of $p \in (1,\infty)$ we must show that if $\phi \in \mathrm{Fam}^{st}_\Gamma(\Lambda_\Gamma, \sigma_\lambda, \varphi)$, then there exists twisting and embedding data such that $i_*\phi \in H^{p,<r_{p,\lambda}(\Gamma)'}(\partial X, V(\sigma'_{\lambda'}, \varphi'))$.

From now on p is called generic if p is irrational and

$$\epsilon := \left[\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho - \delta_\Gamma}{p}\right] - \left[-2 \max_{[P]_\Gamma \in \mathcal{P}} \left[\delta_{\varphi_{|\Gamma_P}} + \rho_{\Gamma_P}\right] + 2\frac{\rho}{p}\right] \neq 0 \ .$$

It is easy to see that the set of generic p is a dense subset of $(1,\infty)$ since either $\rho - \delta_\Gamma \neq 2\rho$ or Γ is elementary hyperbolic and has no cusps.

Since ϕ is stably deformable we can choose twisting and embedding data such that

$$(i_{(1)})_*\phi \in {}^\Gamma C^{-\infty}(\partial X^{(1)}, V^{(1)}(\sigma^{(1)}_{\lambda^{(1)}}, \varphi^{(1)}))$$

is deformable. By twisting further if necessary we can assume that $\sigma^{(1)}$ is the trivial representation $1 \in \hat{M}^{(1)}$ (see [5], (33)). Choosing further embedding data we can in addition assume that

$$(i_{(2)})_*(i_{(1)})_*\phi \in {}^\Gamma C^{-\infty}(\partial X^{(2)}, V^{(2)}(1_{\lambda^{(2)}}, \varphi^{(1)})) \ ,$$

such that

$$\begin{aligned}
\mathrm{Re}(\lambda^{(2)}) &< 0 && (28)\\
\max_{[P]_\Gamma \in \mathcal{P}^{(2)}} \left(\delta_{\varphi^{(1)}_{|\Gamma_P}} - (\rho^{(2)})^{\Gamma_P}\right) &< 0 && (29)\\
r^{(2)}_{p,\lambda^{(2)}}(\Gamma) &< 0 && (30)\\
\frac{\rho^{(2)} + \delta^{(2)}_\Gamma}{\rho^{(2)} - \delta^{(2)}_\Gamma} &< |\frac{\epsilon}{\alpha}| && (31)\\
(\rho^{(2)} - \delta^{(2)}_\Gamma) \min\left(\frac{1}{p}, \frac{1}{q}\right) &> \alpha \ , && (32)
\end{aligned}$$

and

$$\min\left(-\mathrm{Re}(\lambda^{(2)}) - \rho^{(2)} - \delta_{\varphi^{(2)}} + \frac{\rho^{(2)} - \delta^{(2)}_\Gamma}{p}, -2 \max_{[P]_\Gamma \in \mathcal{P}^{(2)}} \left[\delta_{\varphi^{(2)}_{|\Gamma_P}} + \rho^{(2)}_{\Gamma_P}\right] + 2\frac{\rho^{(2)}}{p}\right) > 0 \ . \quad (33)$$

Let d denote the codimension of the embedding. Then $\lambda^{(2)} = \lambda^{(1)} - \frac{1}{2}d\alpha$, $\rho^{(2)} = \rho^{(1)} + \frac{1}{2}d\alpha$, $(\rho^{(2)})^{\Gamma_P} = (\rho^{(1)})^{\Gamma_P} + \frac{1}{2}d\alpha$, and $\delta^{(2)}_\Gamma = \delta^{(1)}_\Gamma - \frac{1}{2}d\alpha$. So we can get (28) for sufficiently large d.

Moreover, we have the relations

$$\max_{[P]_\Gamma\in\mathcal{P}^{(2)}}\left(\delta_{\varphi^{(1)}_{|\Gamma_P}}-(\rho^{(2)})^{\Gamma_P}\right)=\max_{[P]_\Gamma\in\mathcal{P}^{(1)}}\left(\delta_{\varphi^{(1)}_{|\Gamma_P}}-(\rho^{(1)})^{\Gamma_P}\right)-\frac{1}{2}d\alpha\ .$$

Thus we can obtain (29) for sufficiently large d. Since $r^{(2)}_{p,\lambda^{(2)}}(\Gamma)=r^{(1)}_{p,\lambda^{(1)}}(\Gamma)-\frac{d}{q}$ we can also get (30). In order to obtain (33) note that embedding in codimension d increases the left hand side by $d\alpha$. For (31) and 32 note that ϵ does not change under embedding and twisting, but $\frac{\rho^{(2)}+\delta^{(2)}_\Gamma}{\rho^{(2)}-\delta^{(2)}_\Gamma}\to 0$ and $\frac{\alpha}{\rho^{(2)}-\delta^{(2)}_\Gamma}\to 0$ as $d\to\infty$. If $\epsilon>0$, then we consider the non-empty interval

$$I^{(1)}:=\left(\left[-2\max_{[P]_\Gamma\in\mathcal{P}^{(1)}}\left[\delta_{\varphi^{(1)}_{|\Gamma_P}}+\rho^{(1)}_{\Gamma_P}\right]+2\frac{\rho^{(1)}}{p}\right],\left[\mathrm{Re}(\lambda^{(1)})-\rho^{(1)}-\delta^{(1)}_\varphi+\frac{\rho^{(1)}-\delta^{(1)}_\Gamma}{p}\right]\right)\ .$$

If $\epsilon<0$, then we set

$$I^{(1)}:=\left(\left[\mathrm{Re}(\lambda^{(1)})-\rho^{(1)}-\delta^{(1)}_\varphi+\frac{\rho^{(1)}-\delta^{(1)}_\Gamma}{p}\right],\left[-2\max_{[P]_\Gamma\in\mathcal{P}^{(1)}}\left[\delta_{\varphi^{(1)}_{|\Gamma_P}}+\rho^{(1)}_{\Gamma_P}\right]+2\frac{\rho^{(1)}}{p}\right]\right)\ .$$

Then $I^{(2)}=I^{(1)}+\frac{d\alpha}{p}$. Since p is irrational there are arbitrary large d such that $I^{(2)}$ contains an integer. We now fix d such that the inequalities above are satisfied and $I^{(2)}$ contains an integer $k\in\mathbb{N}$.

In order to simplify the notation we now replace ϕ by $(i_{(2)})_*(i_{(1)})_*\phi$, and we omit the superscripts $()^{(2)}$ everywhere. Thus we can assume that

$$\begin{array}{rcl}
\phi & \in & \mathrm{Fam}_\Gamma(\Lambda_\Gamma,1_\lambda,\varphi)\\
\mathrm{Re}(\lambda) & < & 0\\
\max_{[P]_\Gamma\in\mathcal{P}}\left(\delta_{\varphi_{|\Gamma_P}}-\rho^{\Gamma_P}\right) & < & 0\\
r_{p,\lambda}(\Gamma) & < & 0\\
\min\left(-\mathrm{Re}(\lambda)-\rho-\delta_\varphi+\frac{\rho-\delta_\Gamma}{p}\ ,\ -2\max_{[P]_\Gamma\in\mathcal{P}}\left[\delta_{\varphi_{|\Gamma_P}}+\rho_{\Gamma_P}\right]+2\frac{\rho}{p}\right) & > & 0\\
\frac{\rho+\delta_\Gamma}{\rho-\delta_\Gamma} & < & |\frac{\epsilon}{\alpha}|\\
(\rho-\delta_\Gamma)\min\left(\frac{1}{p}\ ,\ \frac{1}{q}\right) & > & \alpha,
\end{array}$$

and I contains an integer k.

We must show that $\phi\in H^{p,<r_{p,\lambda}(\Gamma)}$. Since ϕ is deformable and strongly supported on the limit set we can find a germ at λ of a holomorphic family $(\phi_\mu)_\mu$, $\phi_\mu\in {}^\Gamma C^{-\infty}(\partial X,V(1_\mu,\varphi))$, such that $\phi_\lambda=\phi$ and $(res^\Gamma(\phi_\mu))_{|\mu=\lambda}=0$. By Lemma 2.12 we have that $\phi\in\mathrm{Ext}^{sing}_\Gamma(1_\lambda,\varphi)$. Since ext^Γ has

at most finite-dimensional singularities and $B_\Gamma(1_\lambda,\varphi)\subset D_\Gamma(1_\lambda,\varphi)$ is dense we can find a germ of a holomorphic family $(\psi_\mu)_\mu$, $\psi_\mu\in B_\Gamma(1_\mu,\varphi)$, such that $\psi_\lambda=0$ and $(ext^\Gamma\psi_\mu)_{|\mu=\lambda}=\phi$. We now define the germ at $-\lambda$ of a meromorphic family $(f_\mu)_\mu$, $f_\mu\in B_\Gamma(1_\mu,\varphi)$, by $f_{-\mu}:=\hat{S}^{w,\Gamma}_{1_\mu,\varphi}\psi_\mu$. Here $\hat{S}^{w,\Gamma}_{1_\mu,\varphi}:D_\Gamma(1_\mu,\varphi)\to D_\Gamma(1_{-\mu},\varphi)$ (note that $1^w=1$) is the scattering matrix defined in 2.14, and we employ Lemma 2.15 in order to see that $f_\mu\in B_\Gamma(1_\mu,\varphi)$. Let $p_1(\lambda)$ be the Plancherel density. It is a meromorphic function on $\mathfrak{a}^*_\mathbb{C}$. We then have the following identities:

$$\begin{aligned} ext^\Gamma_{-\mu}\circ\hat{S}^{w,\Gamma}_{1_\mu,\varphi} &= \hat{J}^w_{1_\mu,\varphi}\circ ext^\Gamma_\mu\\ p_1(-\mu)\hat{J}^{w^{-1}}_{1_{-\mu},\varphi}\circ\hat{J}^w_{1_\mu,\varphi} &= \mathrm{id}\ . \end{aligned}$$

Thus we can write

$$\phi=\left(\frac{1}{p_1(-\mu)}\hat{J}^{w^{-1}}_{1_{-\mu},\varphi}(ext^\Gamma f_{-\mu})\right)_{|\mu=\lambda}\ .$$

Let $\hat{f}_\mu:=\sum_k(\frac{\mu+\lambda}{\alpha})^k\hat{f}_k$ be the Laurent expansion of the germ $(ext^\Gamma f_\mu)_\mu$ at $-\lambda$. Then by Corollary 4.12 we have $\hat{f}_k\in H^{p,<r'}$, where r' is determined by

$$r'\alpha:=\min\left(-\mathrm{Re}(\lambda)-\rho-\delta_\varphi+\frac{\rho-\delta_\Gamma}{p}\ ,\ -2\max_{[P]_\Gamma\in\mathcal{P}}\left[\delta_{\varphi_{|\Gamma_P}}+\rho_{\Gamma_P}\right]+2\frac{\rho}{p}\right)\ .$$

Let

$$\frac{1}{p_1(\mu)}\hat{J}^{w^{-1}}_{1_\mu,\varphi}=\sum_l(\frac{\mu+\lambda}{\alpha})^l B_l$$

be the Laurent expansion of $(\frac{1}{p_1(\mu)}\hat{J}^{w^{-1}}_{1_\mu,\varphi})_\mu$ at $-\lambda$. Since $\mathrm{Re}(\lambda)<0$ we can apply Cor. 3.8 in order to get $B_l:H^{p,<r'}\to H^{p,<r'+2\frac{\mathrm{Re}(\lambda)}{\alpha}}$. Since $r'+2\frac{\mathrm{Re}(\lambda)}{\alpha}=r_{p,\lambda}(\Gamma)$ and $\phi=\sum_{l+k=0}B_l(\hat{f}_k)$ we conclude that $\phi\in H^{p,<r_{p,\lambda}(\Gamma)}$.

Now we show the second assertion of Theorem 4.3. Let $\phi\in\mathrm{Cusp}_\Gamma(\sigma_\lambda,\varphi)$ and $\lambda\not\in I_\mathfrak{a}$. Again, after twisting and embedding we can assume that all cusps of Γ have smaller rank,

$$\begin{aligned} \phi &\in \mathrm{Cusp}_\Gamma(1_\lambda,\varphi)\cap\mathrm{Fam}_\Gamma(\Lambda_\Gamma,1_\lambda,\varphi)\\ \mathrm{Re}(\lambda) &< 0\\ r^0_{p,\lambda}(\Gamma) &< 0\\ -\mathrm{Re}(\lambda)-\rho-\delta_\varphi+\frac{\rho-\delta_\Gamma}{p} &> 0\\ (\rho-\delta_\Gamma)\min\left(\frac{1}{p}\ ,\ \frac{1}{q}\right) &> \alpha. \end{aligned}$$

Let $(f_\mu)_\mu$, $f_\mu\in B_\Gamma(1_\mu,\varphi)$, be constructed as above. Because we assume $\lambda\not\in I_\mathfrak{a}$ this family is regular at $\mu=-\lambda$.

We show that $f_{-\lambda} \in S_\Gamma(1_{-\lambda}, \varphi)$. Let $[P]_\Gamma \in \mathcal{P}$. We choose a P-invariant the cut-off function χ_P in order to define the map T_P (see Subsection 2.1). Then we can define the constant term

$$(T_P f_{-\lambda})_P := \int_{\Gamma_P \backslash P_\Gamma} \pi^{1_{-\lambda},\varphi}(x) T_P f_{-\lambda}\, dx \ .$$

Taking the constant term commutes with multiplication by χ_P, restriction and the intertwining operators. Therefore we have

$$\begin{aligned} (T_P f_{-\lambda})_P &= \left(\chi_P res^{\Gamma_P} \hat{J}^w_{1_{-\lambda},\varphi} \phi_\lambda\right)_P \\ &= \chi_P res^{\Gamma_P} \hat{J}^w_{1_{-\lambda},\varphi} (\phi_\lambda)_P \ . \end{aligned}$$

Since ϕ is a cusp form we conclude that $(T_P f_{-\lambda})_P = 0$. Since $AS(T_P f_{-\lambda}) = AS\,(T_P f_{-\lambda})_P = 0$ for all $P \in \mathcal{P}$ we see that $f_{-\lambda} \in S_\Gamma(1_{-\lambda}, \varphi)$.

Using

$$\phi = \left(\frac{1}{p_1(-\lambda)} \hat{J}^{w^{-1}}_{1_{-\lambda},\varphi}(ext^\Gamma f^{mod}_{-\lambda})\right)$$

and Corollary 4.11, 2.) we obtain $ext^\Gamma f_{-\lambda} = \widehat{f}_{-\lambda} \in H^{p,<r'}$ for $r'\alpha := -\mathrm{Re}(\lambda) - \rho - \delta_\varphi + \frac{\rho - \delta_\Gamma}{p}$. It follows from Corollary 3.8 and $r^0_{p,\lambda}(\Gamma) = r' + 2\frac{\mathrm{Re}(\lambda)}{\alpha}$ that $\phi \in H^{p,<r^0_{p,\lambda}(\Gamma)}$. □

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